Graph Cuts and Linear Programming

Topic 2.1: Linear Programming Review

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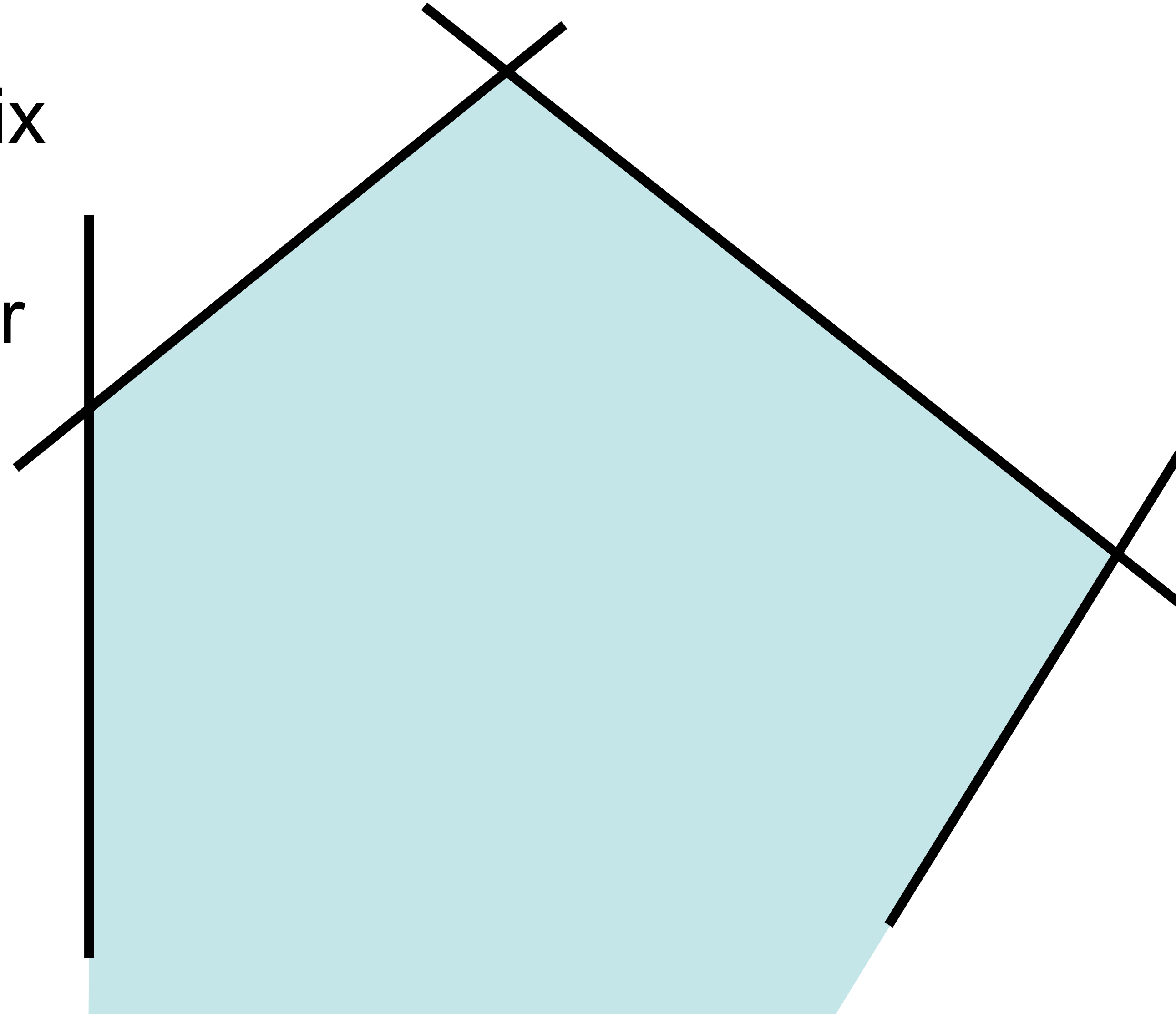
http://www.robots.ox.ac.uk/~oval/

Polyhedron

A$\mathbf{x} \leq \mathbf{b}$

A : m x n matrix

b: m x 1 vector

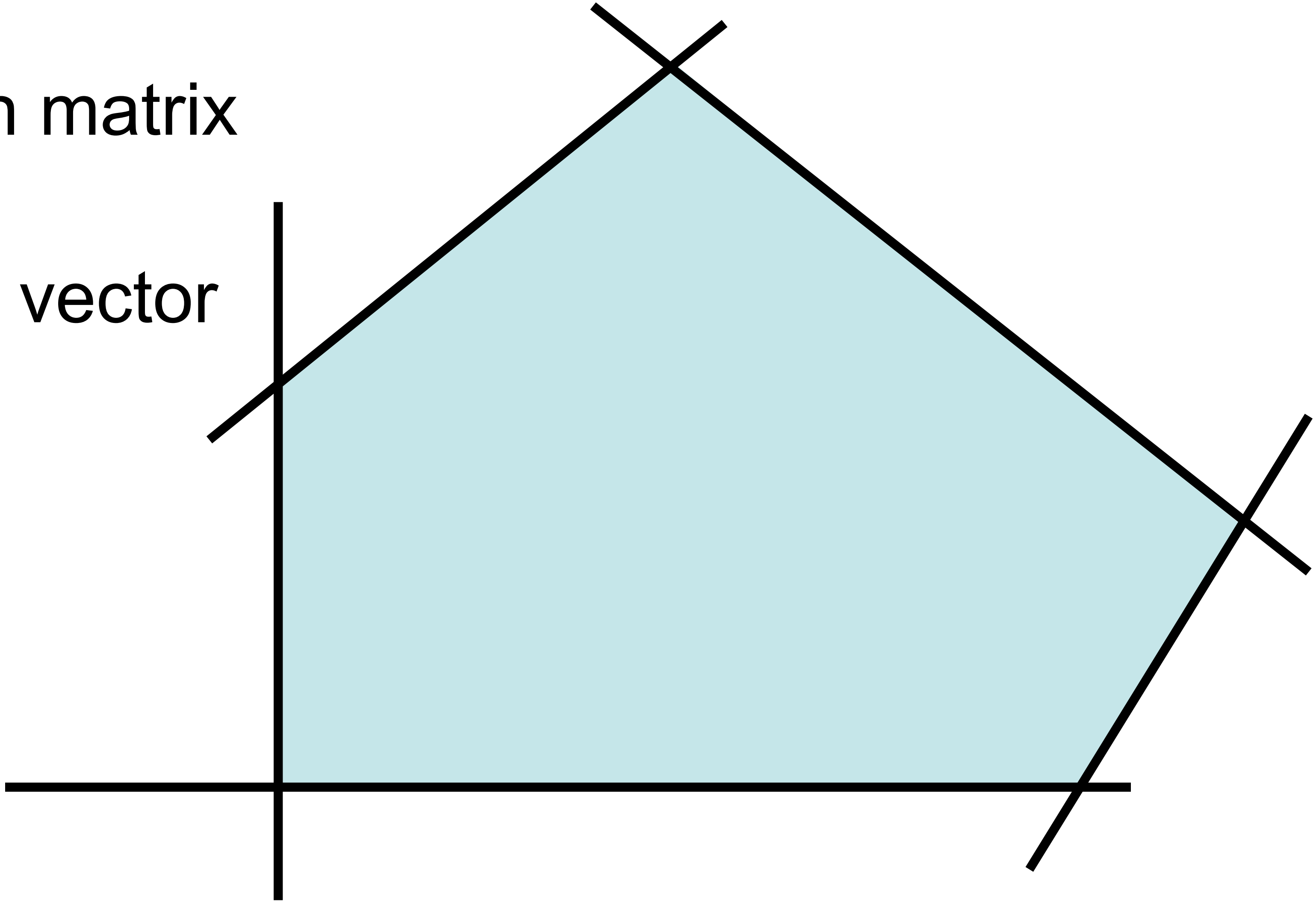


Bounded Polyhedron = Polytope

A**x** ≤ **b**

A : m x n matrix

b: m x 1 vector



Vertex

$\mathbf{z}$ is a vertex of $P = \{\mathbf{x}, A\mathbf{x} \leq \mathbf{b}\}$

$\mathbf{z}$ is not a convex combination of two points in P

There does not exist $\mathbf{x}, \mathbf{y} \in P$ and $0 < \lambda < 1$

$\mathbf{x} \neq \mathbf{z}$ and $\mathbf{y} \neq \mathbf{z}$

such that $\mathbf{z} = \lambda\, \mathbf{x} + (1-\lambda)\, \mathbf{y}$

Vertex

$\mathbf{z}$ is a vertex of $P = \{\mathbf{x}, A\mathbf{x} \leq \mathbf{b}\}$

Recall A is an m x n matrix

$A_{\mathbf{z}}$ is a submatrix of A

Contains all rows of A such that $\mathbf{a}_i^T\mathbf{z} = b_i$

Vertex

z is a vertex of P

$$\Updownarrow$$

Proof?

Rank of $A_{\mathbf{z}}$ = n

See "hidden" slides

Proof Sketch: Necessity

Let $\mathbf{z}$ be a vertex of P

Suppose $\text{rank}(A_{\mathbf{z}}) < n$ $\quad$ $A_{\mathbf{z}}\mathbf{c} = \mathbf{0}$ for some $\mathbf{c} \neq \mathbf{0}$

Then there exists a $d > 0$ such that

$\mathbf{z} - d\mathbf{c} \in P$ $\quad$ $\mathbf{z} + d\mathbf{c} \in P$ $\quad$ **Contradiction**

Proof Sketch: Sufficiency

Suppose $\text{rank}(A_{\mathbf{z}}) = n$ but $\mathbf{z}$ is not a vertex of P

$\mathbf{z} = (\mathbf{x}+\mathbf{y})/2$ for some $\mathbf{x}, \mathbf{y} \in P$, $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}$

For each $\mathbf{a}$ in $A_{\mathbf{z}}$

$\mathbf{a}^T\mathbf{x} \leq b = \mathbf{a}^T\mathbf{z}$ $\quad\quad$ $\mathbf{a}^T\mathbf{y} \leq b = \mathbf{a}^T\mathbf{z}$

Implies $A_{\mathbf{z}}(\mathbf{x}-\mathbf{y}) = \mathbf{0}$ $\quad\quad$ **Contradiction**

Outline

- Linear Programming
- Duality
- Solving the LP

Linear Program

Maximize a linear function

$$\max_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x}$$ Objective function

$$\text{s.t.} \ A\,\mathbf{x} \leq \mathbf{b}$$ Constraints

Over a polyhedral feasible region

A: m x n matrix

b: m x 1 vector

c: n x 1 vector

x: n x 1 vector

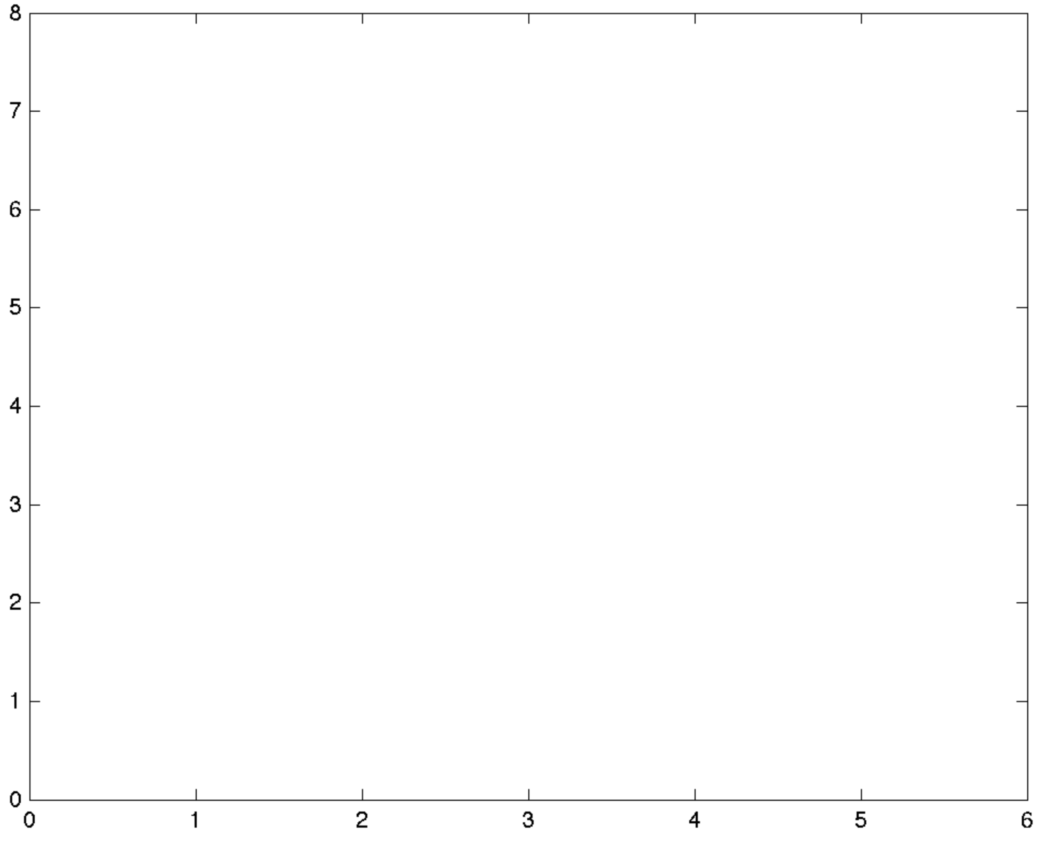
Example

$$\max_{\mathbf{x}}\ x_1 + x_2$$

$$\begin{aligned}
\text{s.t.}\quad & x_1 \geq 0 \\
& x_2 \geq 0 \\
& 4x_1 - x_2 \leq 8 \\
& 2x_1 + x_2 \leq 10 \\
& 5x_1 - 2x_2 \geq -2
\end{aligned}$$

What is c? A? b?

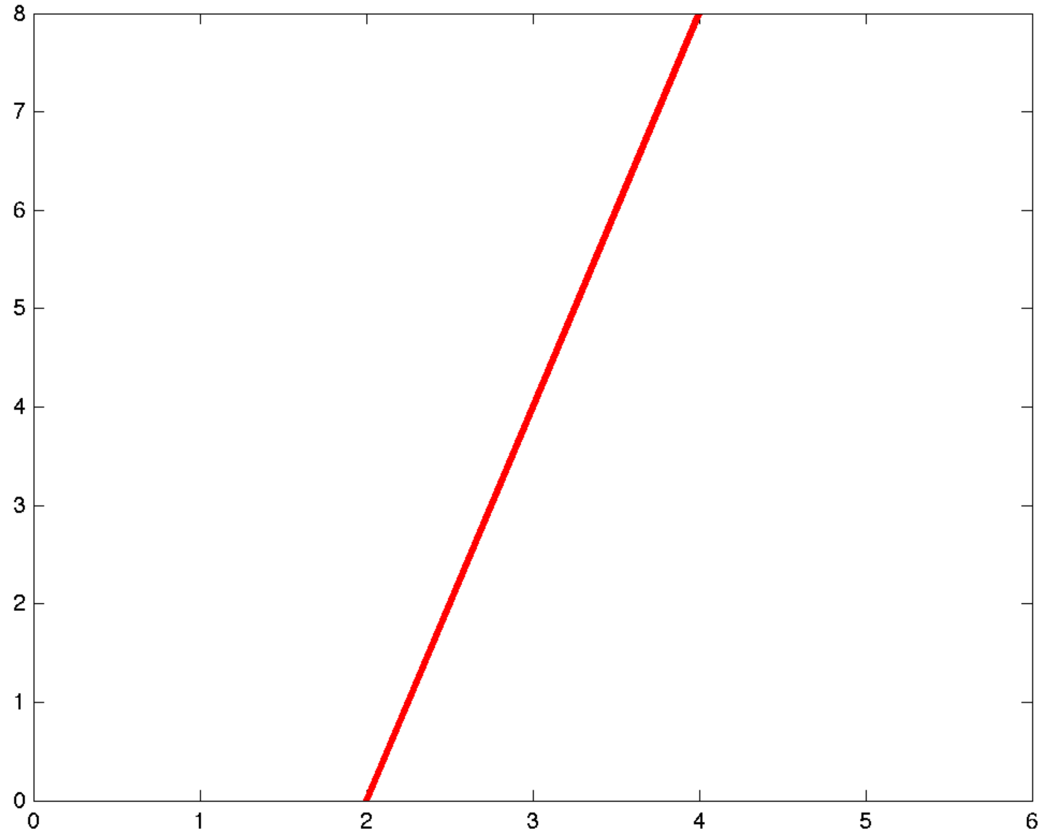
Example



$x_1 \geq 0$

$x_2 \geq 0$

Example

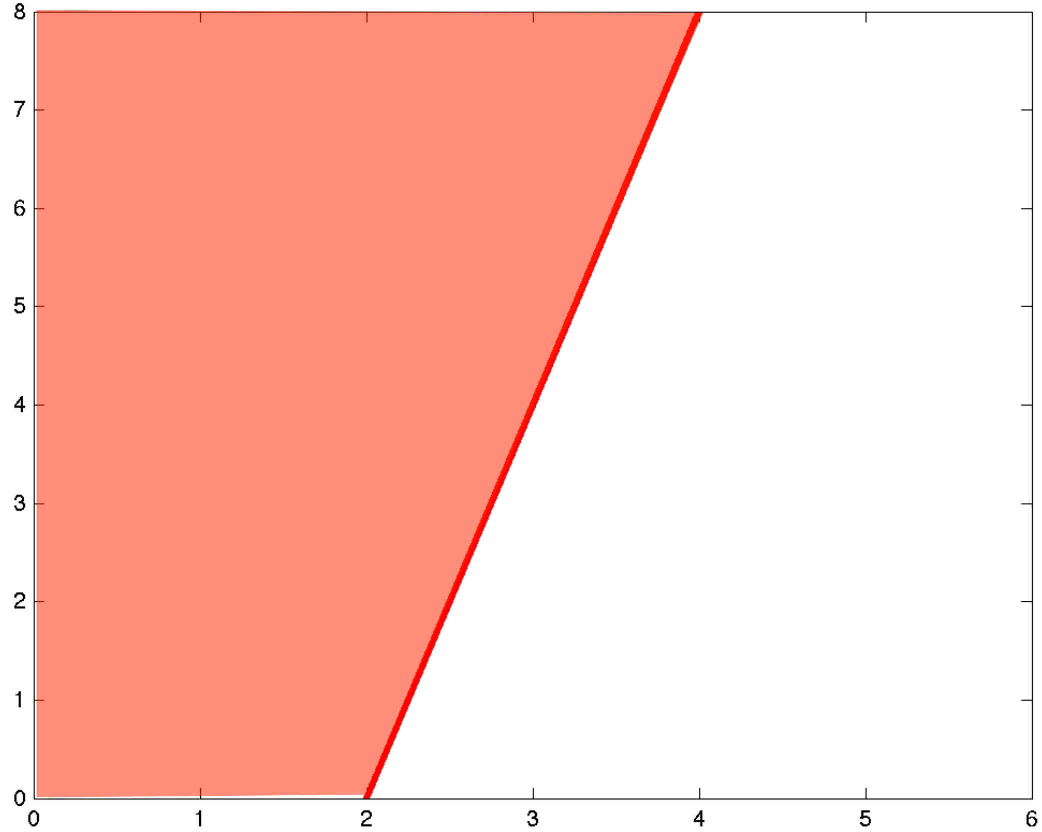


$x_1 \geq 0$

$x_2 \geq 0$

$4x_1 - x_2 = 8$

Example



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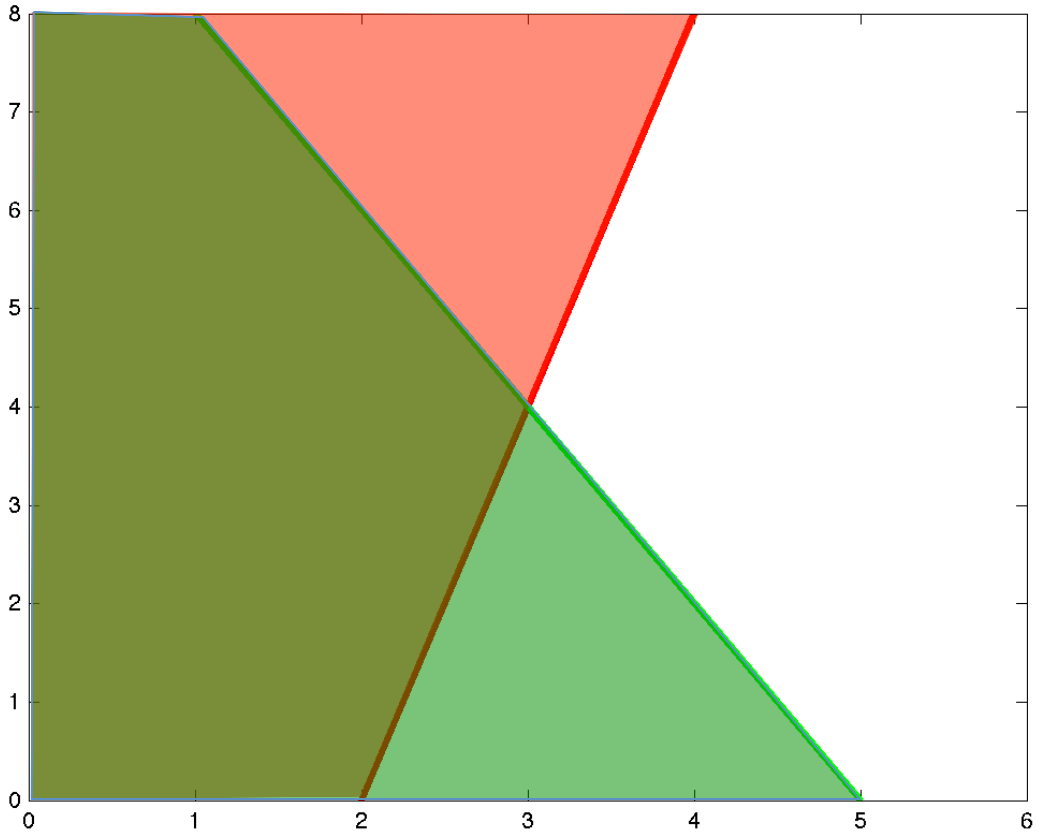

$x_1 \geq 0$

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$4x_1 - x_2 \leq 8$

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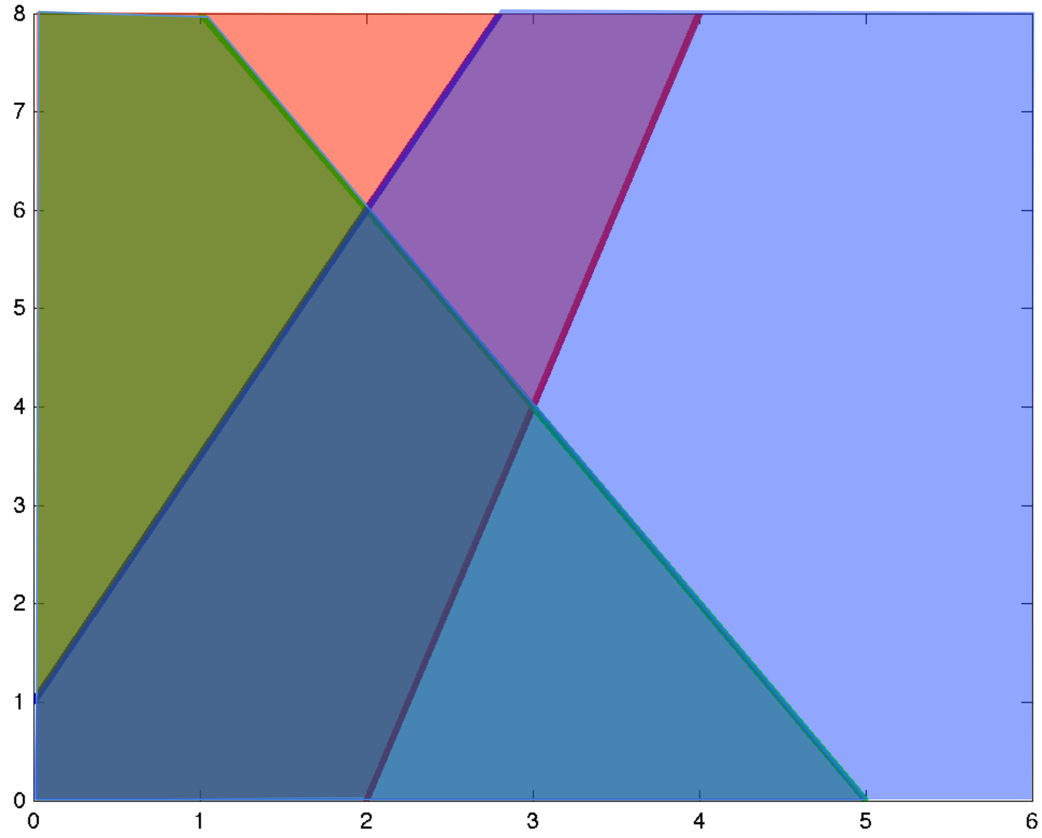
Example



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$2x_1 + x_2 \leq 10$

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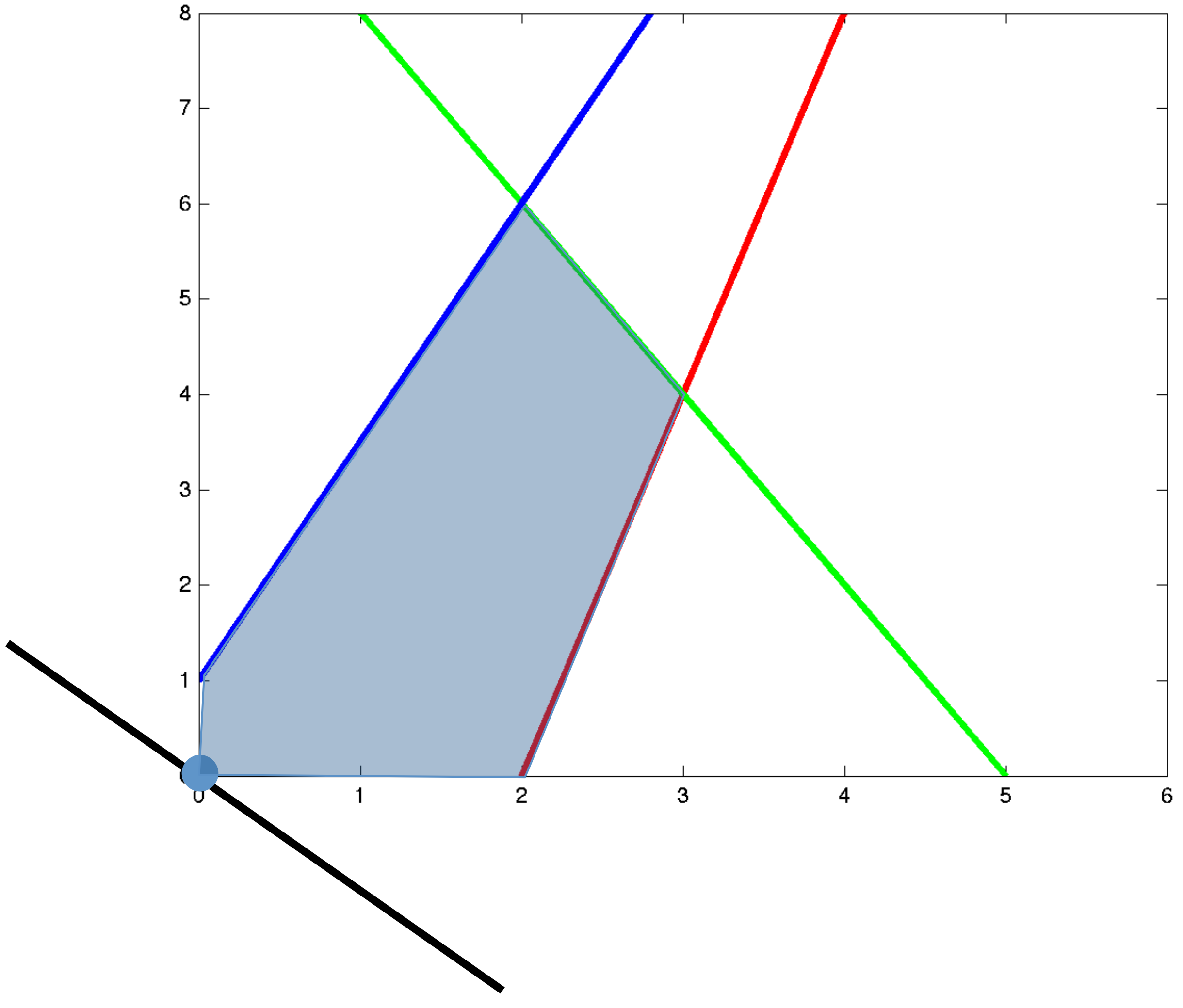
Example



$x_1 \geq 0$

$x_2 \geq 0$

$4x_1 - x_2 \leq 8$

$2x_1 + x_2 \leq 10$

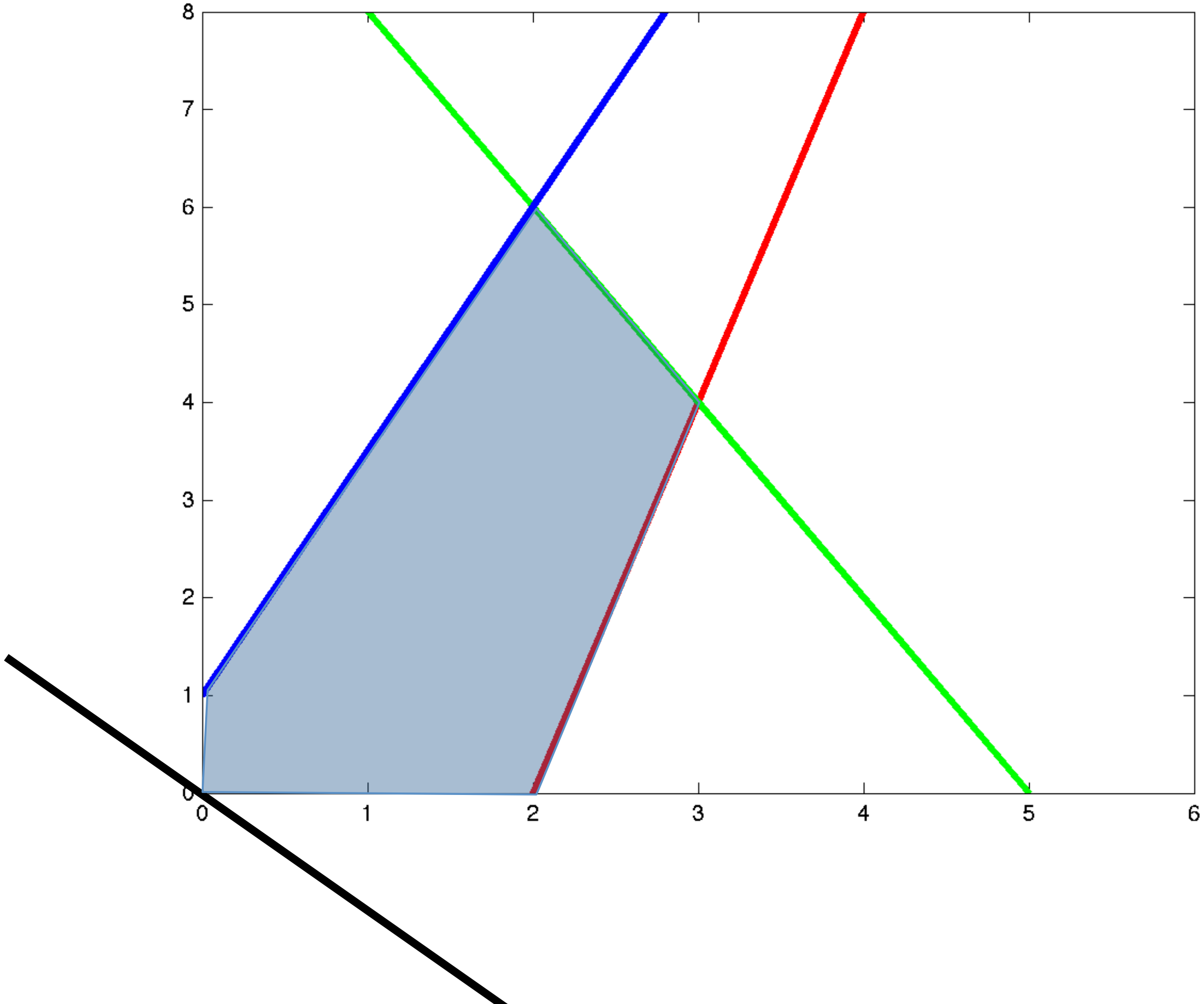
$5x_1 - 2x_2 \geq -2$

$x_1 + x_2 = 0$

$\max_{\mathbf{x}} x_1 + x_2$

Example

$x_1 + x_2 = 8$ **Optimal solution**



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$2x_1 + x_2 \leq 10$

$5x_1 - 2x_2 \geq -2$

$\max_{\mathbf{x}} x_1 + x_2$

Outline

- Linear Programming
- **Duality**
- Solving the LP

Example

$$\max_{\mathbf{x}} 3x_1 + x_2 + 2x_3$$

2 x 7 x

s.t. $-x_1 \leq 0$, $-x_2 \leq 0$, $-x_3 \leq 0$

3 x $x_1 + x_2 + 3x_3 \leq 30$

$2x_1 + 2x_2 + 5x_3 \leq 24$

$4x_1 + x_2 + 2x_3 \leq 36$

Scale the constraints, add them up

$3x_1 + x_2 + 2x_3 \leq 90$ Upper bound on solution

Example

$$\max_{\mathbf{x}} 3x_1 + x_2 + 2x_3$$

1 x

s.t. $-x_1 \leq 0$, $-x_2 \leq 0$, $-x_3 \leq 0$

$x_1 + x_2 + 3x_3 \leq 30$

$2x_1 + 2x_2 + 5x_3 \leq 24$

1 x $4x_1 + x_2 + 2x_3 \leq 36$

Scale the constraints, add them up

$3x_1 + x_2 + 2x_3 \leq 36$ Upper bound on solution

Example

$$\max_{\mathbf{x}} \ 3x_1 + x_2 + 2x_3$$

1 x

s.t. $-x_1 \le 0$, $-x_2 \le 0$, $-x_3 \le 0$

$x_1 + x_2 + 3x_3 \le 30$

$2x_1 + 2x_2 + 5x_3 \le 24$

1 x $4x_1 + x_2 + 2x_3 \le 36$

Scale the constraints, add them up

$3x_1 + x_2 + 2x_3 \le 36$ **Tightest upper bound?**

Example

$$\max_{\mathbf{x}}\ 3x_1 + x_2 + 2x_3$$

$$\begin{array}{ll} & \quad\ \ y_1 \qquad\qquad\quad y_2 \qquad\qquad\quad y_3 \\ \text{s.t.} & -x_1 \le 0,\ -x_2 \le 0,\ -x_3 \le 0 \\ y_4 & x_1 + x_2 + 3x_3 \le 30 \\ y_5 & 2x_1 + 2x_2 + 5x_3 \le 24 \\ y_6 & 4x_1 + x_2 + 2x_3 \le 36 \end{array}$$

We should be able to add up the inequalities

$$y_1, y_2, y_3, y_4, y_5, y_6 \ge 0$$

Example

$\max_{\mathbf{x}}\ 3x_1 + x_2 + 2x_3$

$$\begin{array}{llll} & y_1 & y_2 & y_3 \\ \text{s.t.} & -x_1 \le 0, & -x_2 \le 0, & -x_3 \le 0 \end{array}$$

$y_4 \quad x_1 + x_2 + 3x_3 \le 30$

$y_5 \quad 2x_1 + 2x_2 + 5x_3 \le 24$

$y_6 \quad 4x_1 + x_2 + 2x_3 \le 36$

Coefficient of x_1 should be 3

$$-y_1 + y_4 + 2y_5 + 4y_6 = 3$$

Example

$$\max_{\mathbf{x}}\ 3x_1 + x_2 + 2x_3$$

$$\begin{array}{lccc} & y_1 & y_2 & y_3 \\ \text{s.t.} & -x_1 \le 0, & -x_2 \le 0, & -x_3 \le 0 \end{array}$$

$$\begin{array}{ll} y_4 & x_1 + x_2 + 3x_3 \le 30 \\ y_5 & 2x_1 + 2x_2 + 5x_3 \le 24 \\ y_6 & 4x_1 + x_2 + 2x_3 \le 36 \end{array}$$

Coefficient of x_2 should be 1

$$-y_2 + y_4 + 2y_5 + y_6 = 1$$

Example

$$\max_{\mathbf{x}} 3x_1 + x_2 + 2x_3$$

$$\begin{array}{llll} & y_1 & y_2 & y_3 \\ \text{s.t.} & -x_1 \leq 0, & -x_2 \leq 0, & -x_3 \leq 0 \end{array}$$

$$y_4 \quad x_1 + x_2 + 3x_3 \leq 30$$

$$y_5 \quad 2x_1 + 2x_2 + 5x_3 \leq 24$$

$$y_6 \quad 4x_1 + x_2 + 2x_3 \leq 36$$

Coefficient of x_3 should be 2

$$-y_3 + 3y_4 + 5y_5 + 2y_6 = 2$$

Example

$$\max_{\mathbf{x}}\ 3x_1 + x_2 + 2x_3$$

$$\begin{array}{lccc} & y_1 & y_2 & y_3 \\ \text{s.t.} & -x_1 \le 0, & -x_2 \le 0, & -x_3 \le 0 \end{array}$$

$$\begin{array}{ll} y_4 & x_1 + x_2 + 3x_3 \le 30 \\ y_5 & 2x_1 + 2x_2 + 5x_3 \le 24 \\ y_6 & 4x_1 + x_2 + 2x_3 \le 36 \end{array}$$

Upper bound should be tightest

$$\min_{\mathbf{y}}\ 30y_4 + 24y_5 + 36y_6$$

Dual

$$\min_y\ 30y_4 + 24y_5 + 36y_6$$

$$\text{s.t.}\quad y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$$

$$-y_1 + y_4 + 2y_5 + 4y_6 = 3$$

$$-y_2 + y_4 + 2y_5 + y_6 = 1$$

$$-y_3 + 3y_4 + 5y_5 + 2y_6 = 2$$

Original problem is called primal

Dual of dual is primal

Dual

$$\max_{\mathbf{x}}\ \mathbf{c}^{T}\mathbf{x}$$

$$\text{s.t. } A\,\mathbf{x} \le \mathbf{b}$$

Dual

$$\min_{\mathbf{y} \geq \mathbf{0}} \left[\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} - \mathbf{y}^T(A\,\mathbf{x} - \mathbf{b}) \right]$$

KKT Condition? $A^T\mathbf{y} = \mathbf{c}$

$$\min_{\mathbf{y} \geq \mathbf{0}} \quad \mathbf{b}^T\mathbf{y}$$

$$\text{s.t.} \quad A^T\mathbf{y} = \mathbf{c}$$

$$\max_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x}$$

$$\text{s.t. } A\,\mathbf{x} \le \mathbf{b}$$

Primal

$$\min_{\mathbf{y}\geq\mathbf{0}} \ \mathbf{b}^T\mathbf{y}$$

$$\text{s.t. } A^T\mathbf{y} = \mathbf{c}$$

Dual

Strong Duality

$$\mathbf{p} = \boxed{\begin{array}{l} \max_{\mathbf{x}}\ \mathbf{c}^T\mathbf{x} \\ \\ \text{s.t. } A\,\mathbf{x} \le \mathbf{b} \end{array}}$$

Primal

If $p \neq \infty$ or $d \neq \infty$, then $p = d$.

Think back to the intuition of dual

$$\mathbf{d} = \boxed{\begin{array}{l} \min_{\mathbf{y} \geq \mathbf{0}}\ \ \mathbf{b}^T\mathbf{y} \\ \\ \text{s.t. }\ \ A^T\mathbf{y} = \mathbf{c} \end{array}}$$

Dual

Skipping the proof

Question

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$

$$\text{s.t. } A_1 \mathbf{x} \leq \mathbf{b}_1$$

$$A_2 \mathbf{x} \geq \mathbf{b}_2$$

$$A_3 \mathbf{x} = \mathbf{b}_3$$

Dual?

Weak Duality

Let **x** be a feasible solution of the primal

Let **y** be a feasible solution of the dual

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

Objective of any primal ≤ Objective of any dual

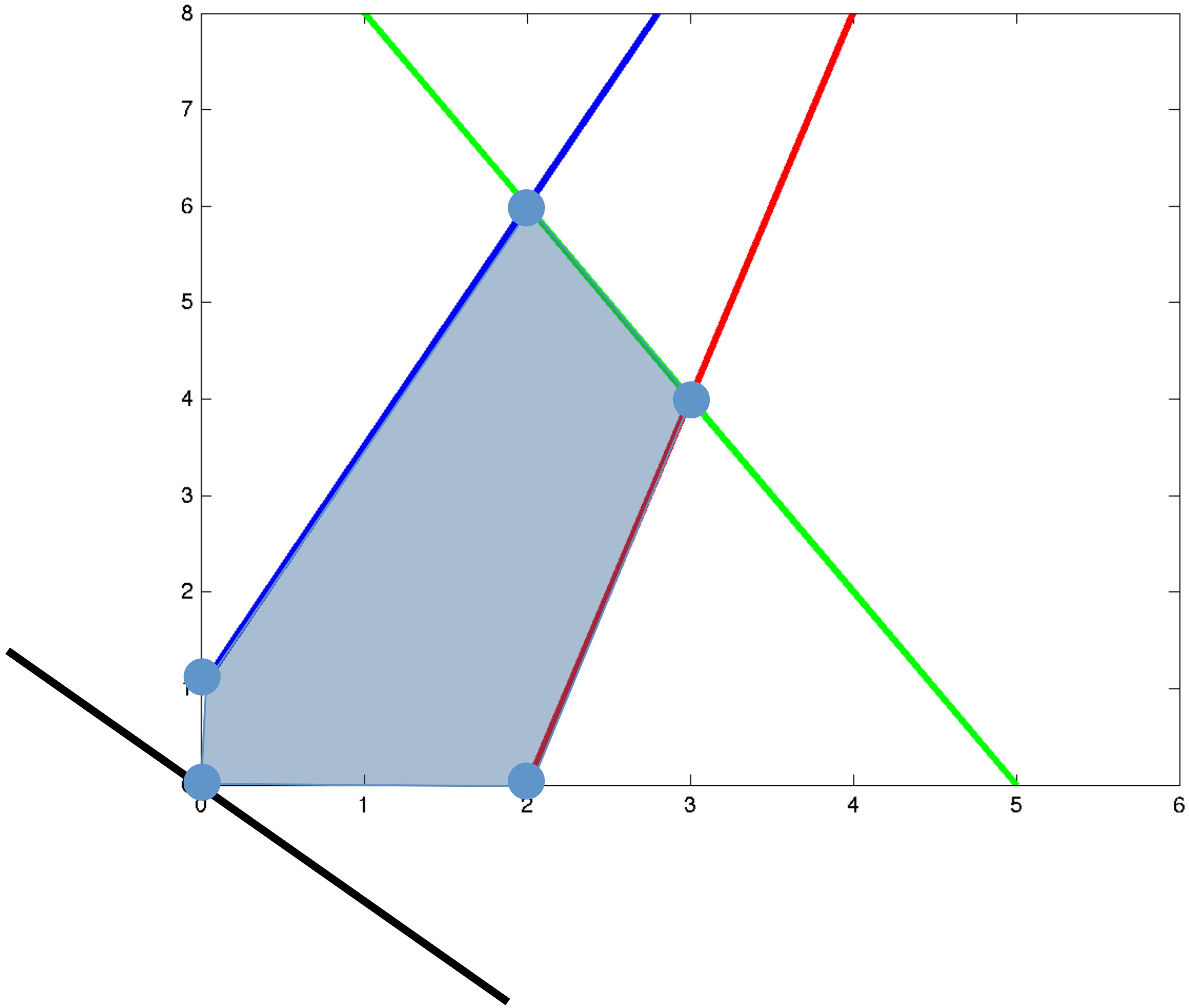
Think back to the intuition of dual

Outline

- Linear Programming
- Duality
- **Solving the LP**

Graphical Solution

$x_1 + x_2 = 8$



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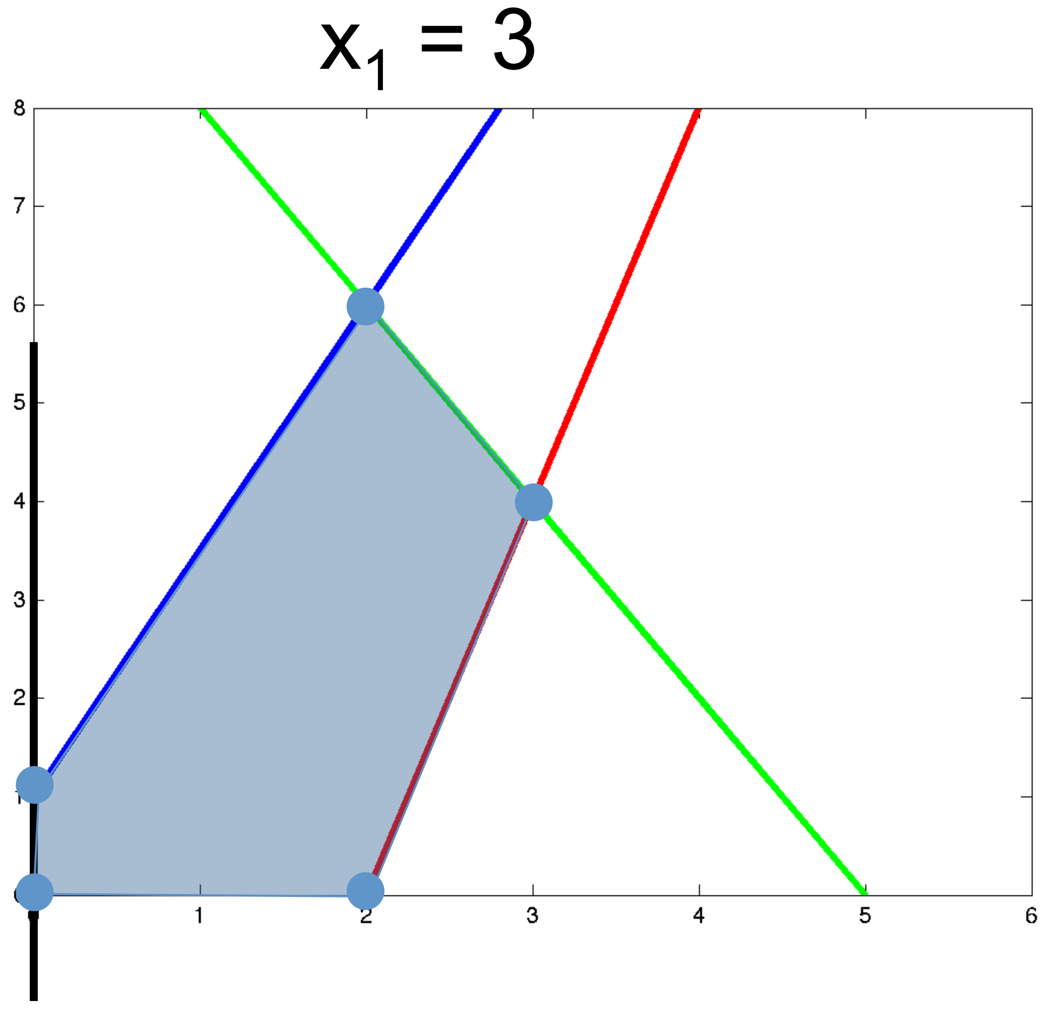
Optimal solution at a vertex

$\max_{\mathbf{x}} x_1 + x_2$

Graphical Solution



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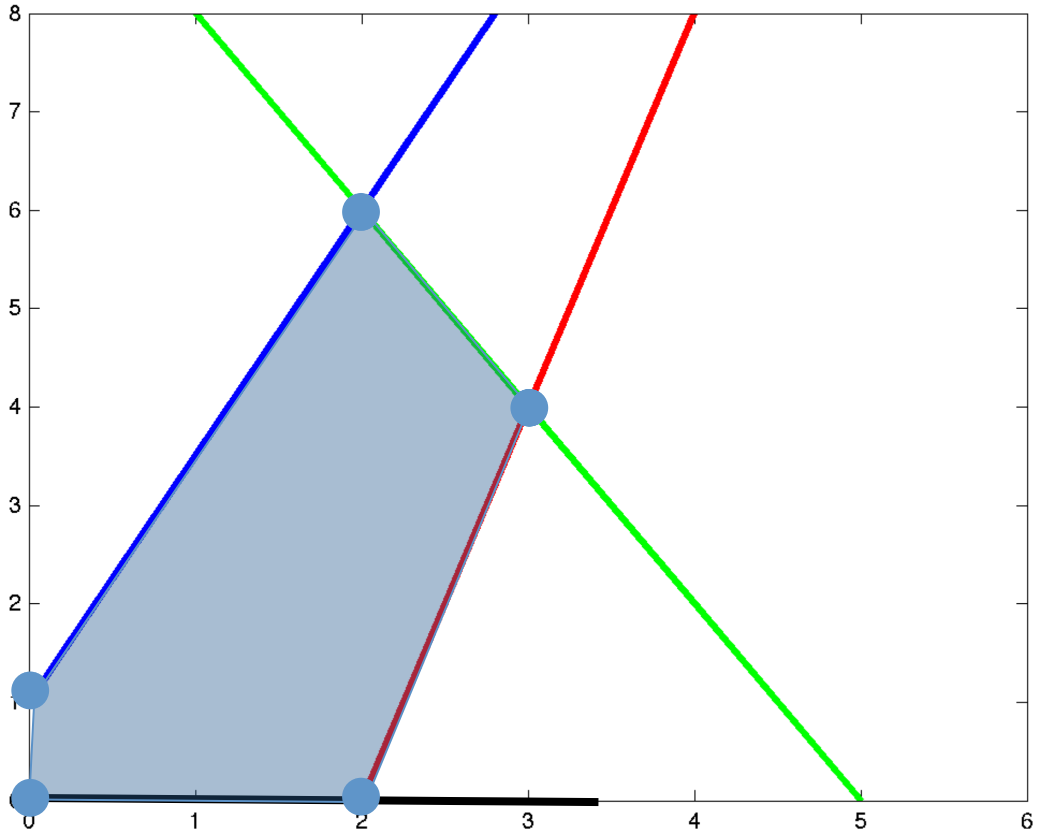
Optimal solution at a vertex

$\max_{\mathbf{x}} x_1$

Graphical Solution

$x_2 = 6$



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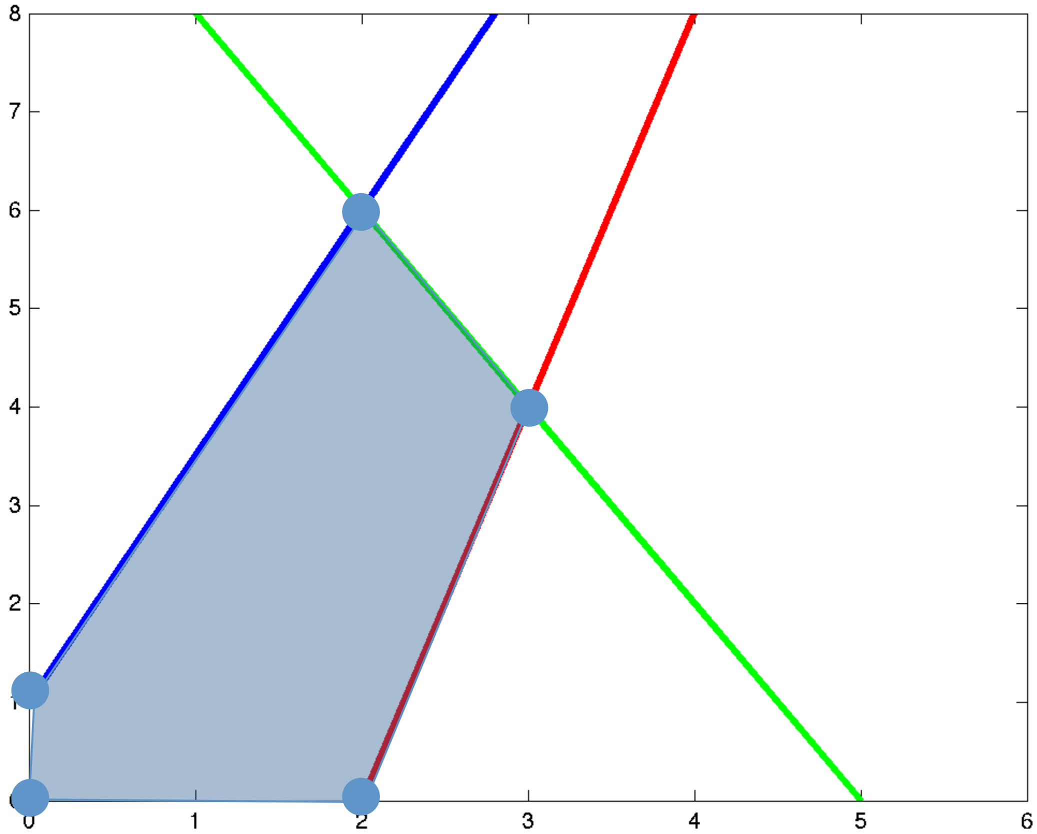
Optimal solution at a vertex

$\max_{\mathbf{x}} x_2$

Graphical Solution



$x_1 \geq 0$

$x_2 \geq 0$

$4x_1 - x_2 \leq 8$

$2x_1 + x_2 \leq 10$

$5x_1 - 2x_2 \geq -2$

An optimal solution always at a vertex

Proof?

$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$

Solving the LP

- Dantzig (1951): Simplex Method
  - Search over vertices of the polyhedra
  - Worst-case complexity is exponential
  - Smoothed complexity is polynomial
- Khachiyan (1979, 1980): Ellipsoid Method
  - Polynomial time complexity
  - LP is a P optimization problem
- Karmarkar (1984): Interior-point Method
  - Polynomial time complexity
  - Competitive with Simplex Method

Solving the LP

- Plenty of standard software available

- Mosek (http://www.mosek.com)
  - C++ API
  - Matlab API
  - Python API
  - Free academic license

Solving the LP

- Dantzig (1951): Simplex Method
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Optimization vs. Feasibility

$$\boxed{\begin{array}{l} \max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \\ \\ \text{s.t. } A\,\mathbf{x} \leq \mathbf{b} \end{array}}$$

Optimization

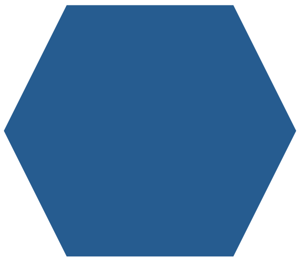
Feasibility asks if there exists an **x** such that

$\mathbf{c}^T\mathbf{x} \geq K$ For a given K

$A\,\mathbf{x} \leq \mathbf{b}$ Feasible solution

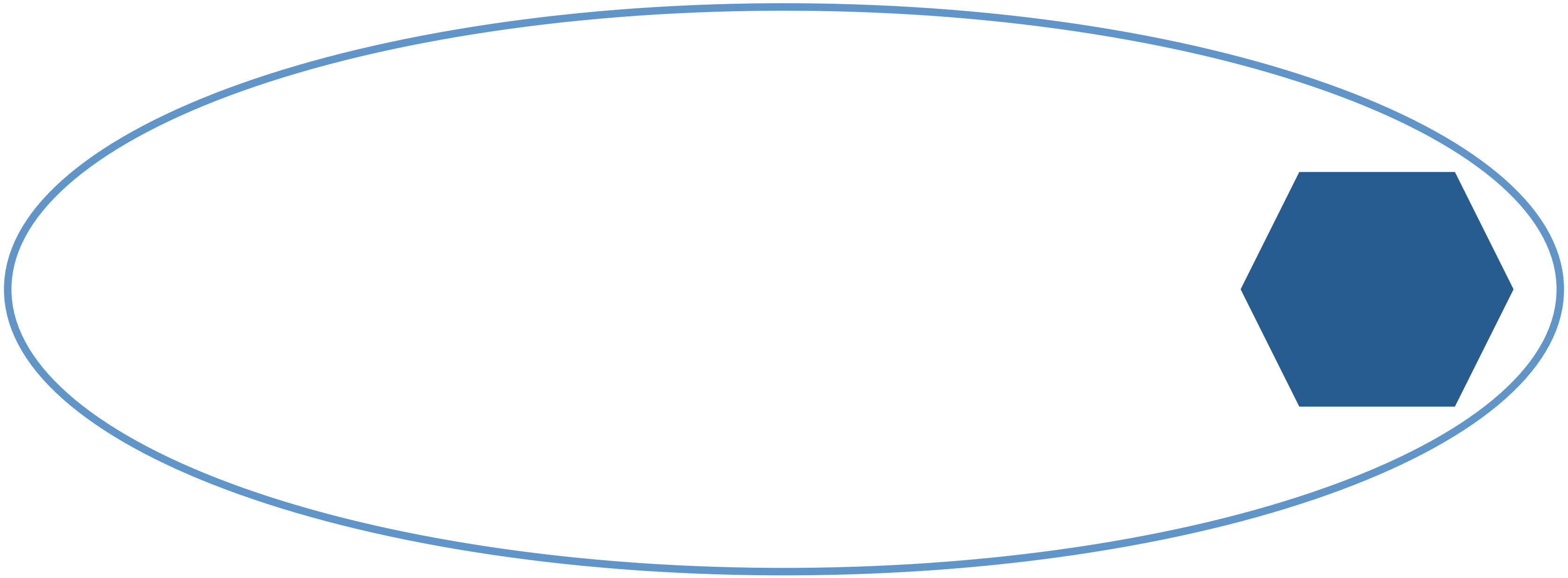
Optimization via binary search on K

Feasibility via Ellipsoid Method



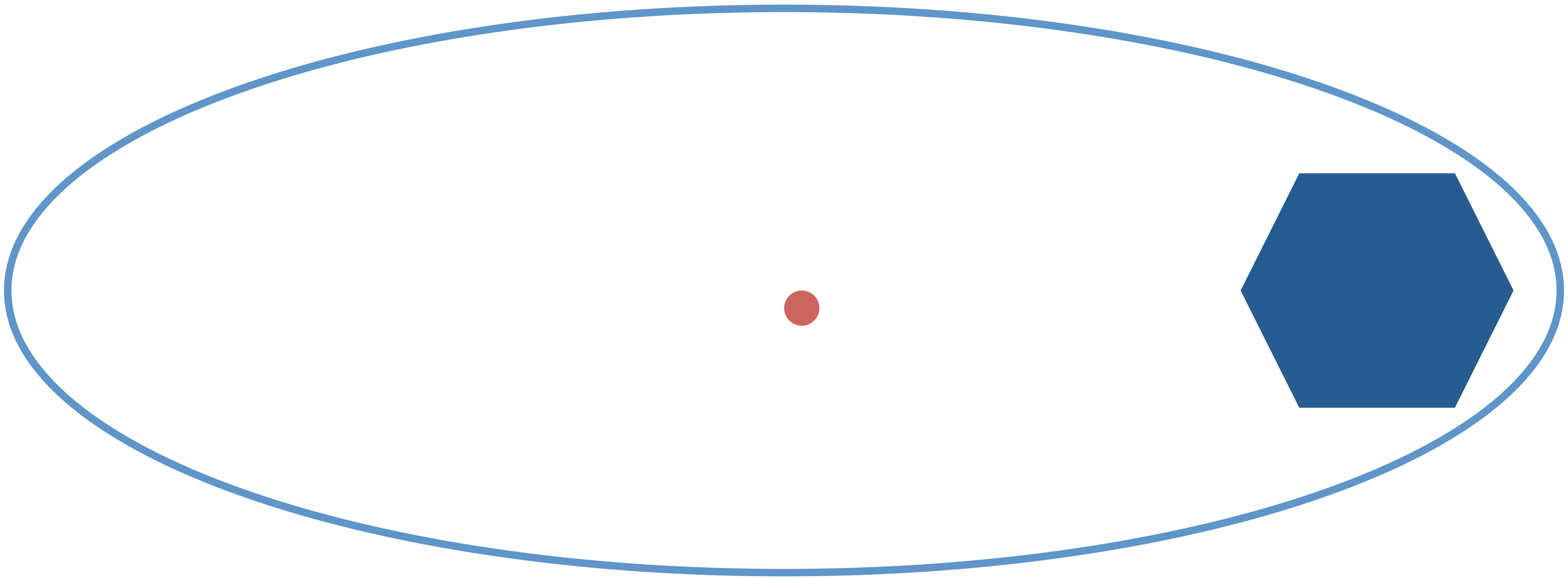
Feasible region of LP

Feasibility via Ellipsoid Method



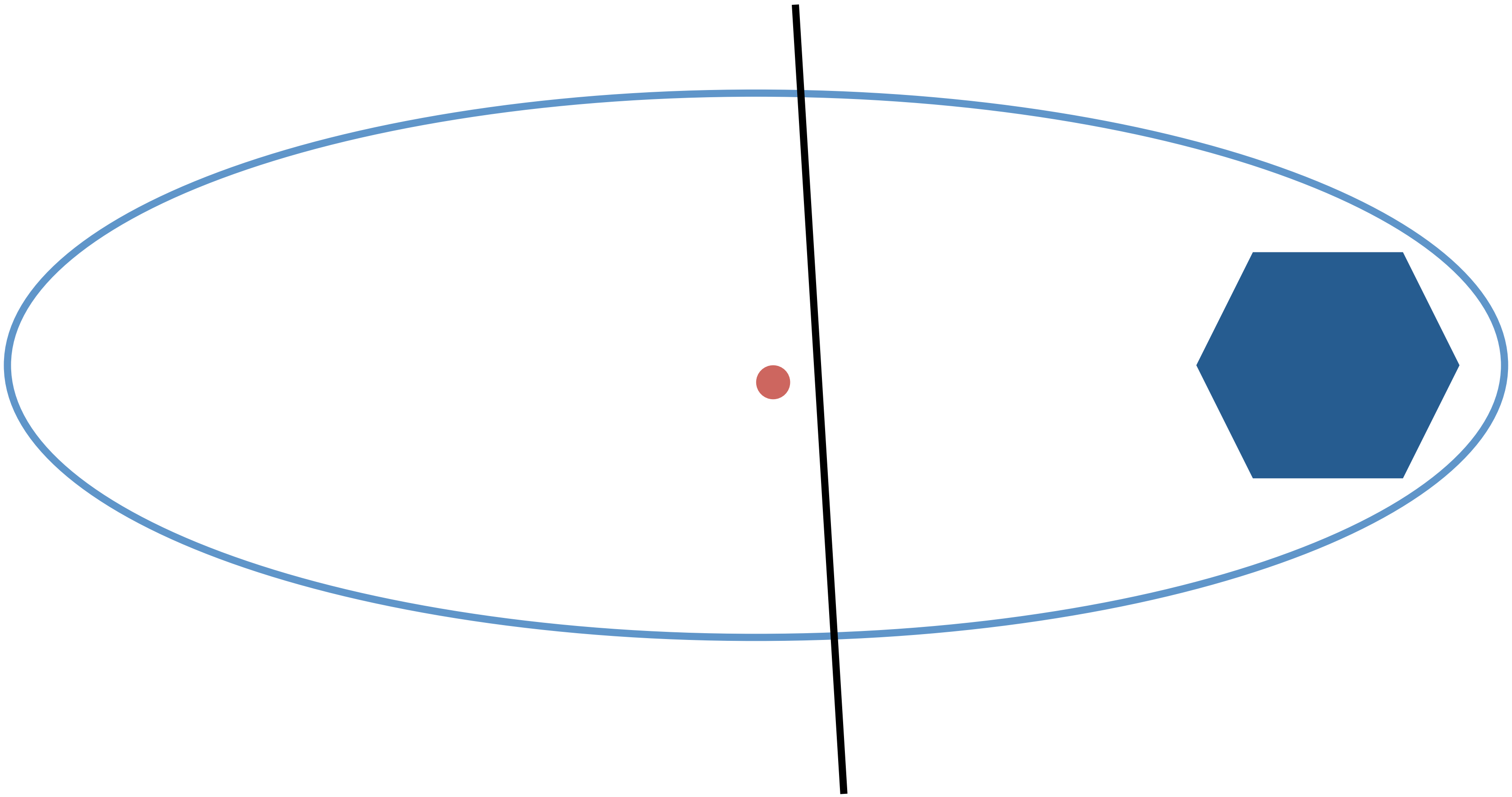
Ellipsoid containing feasible region of LP

Feasibility via Ellipsoid Method



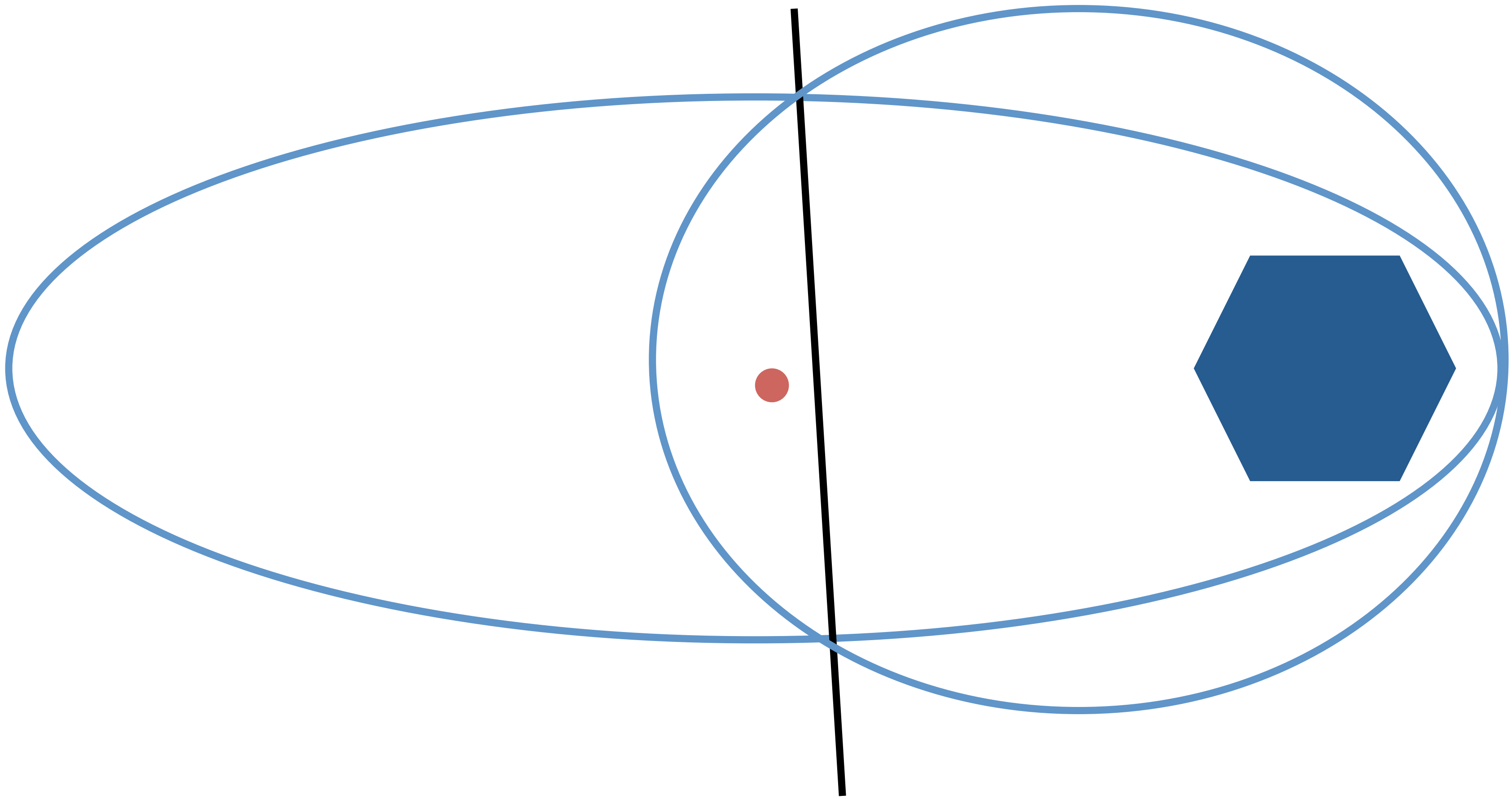
Centroid of ellipsoid

Feasibility via Ellipsoid Method



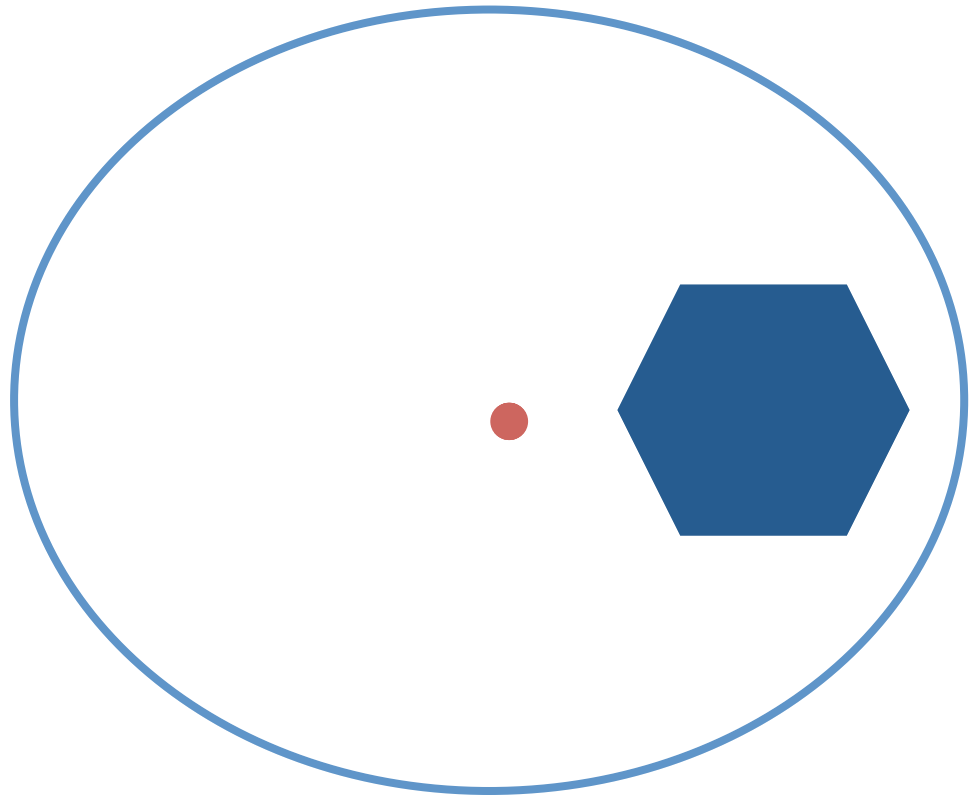
Separating hyperplane for centroid

Feasibility via Ellipsoid Method



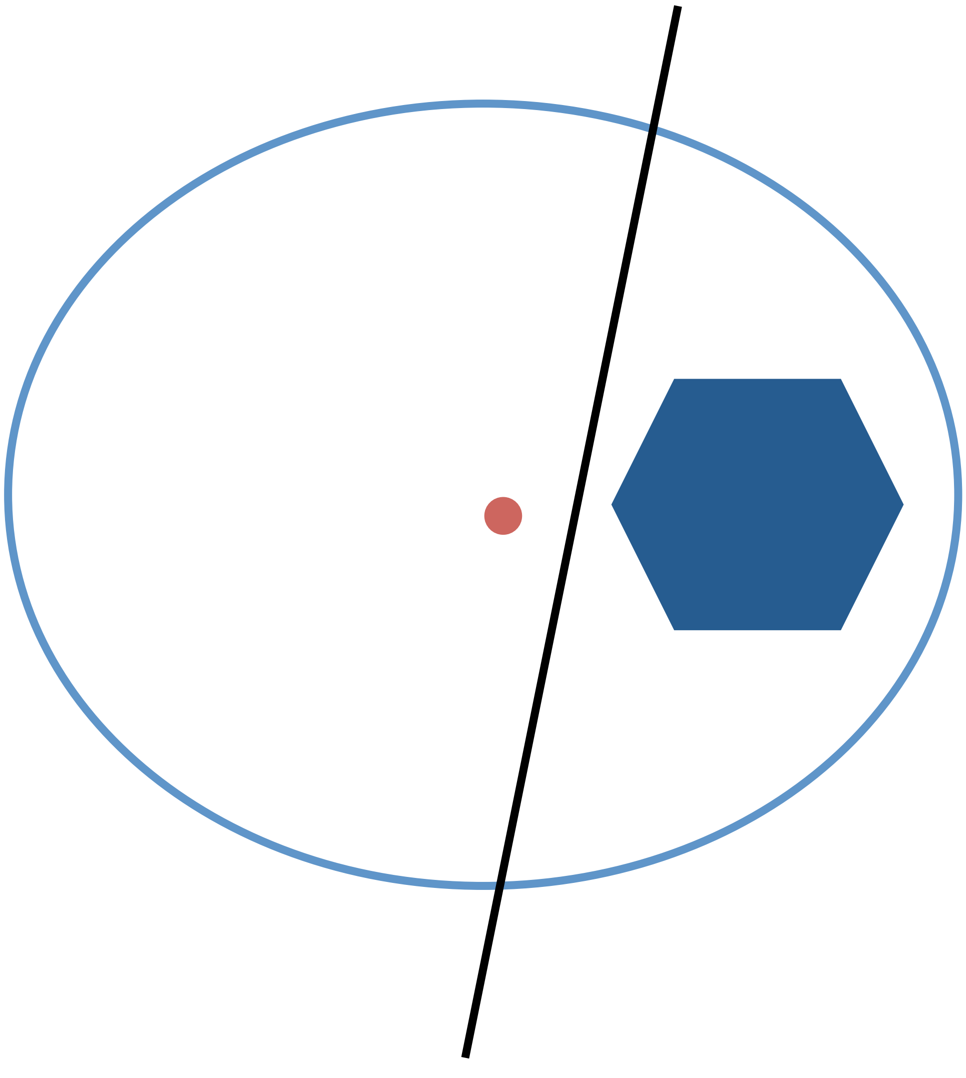
Smallest ellipsoid containing “truncated” ellipsoid

Feasibility via Ellipsoid Method



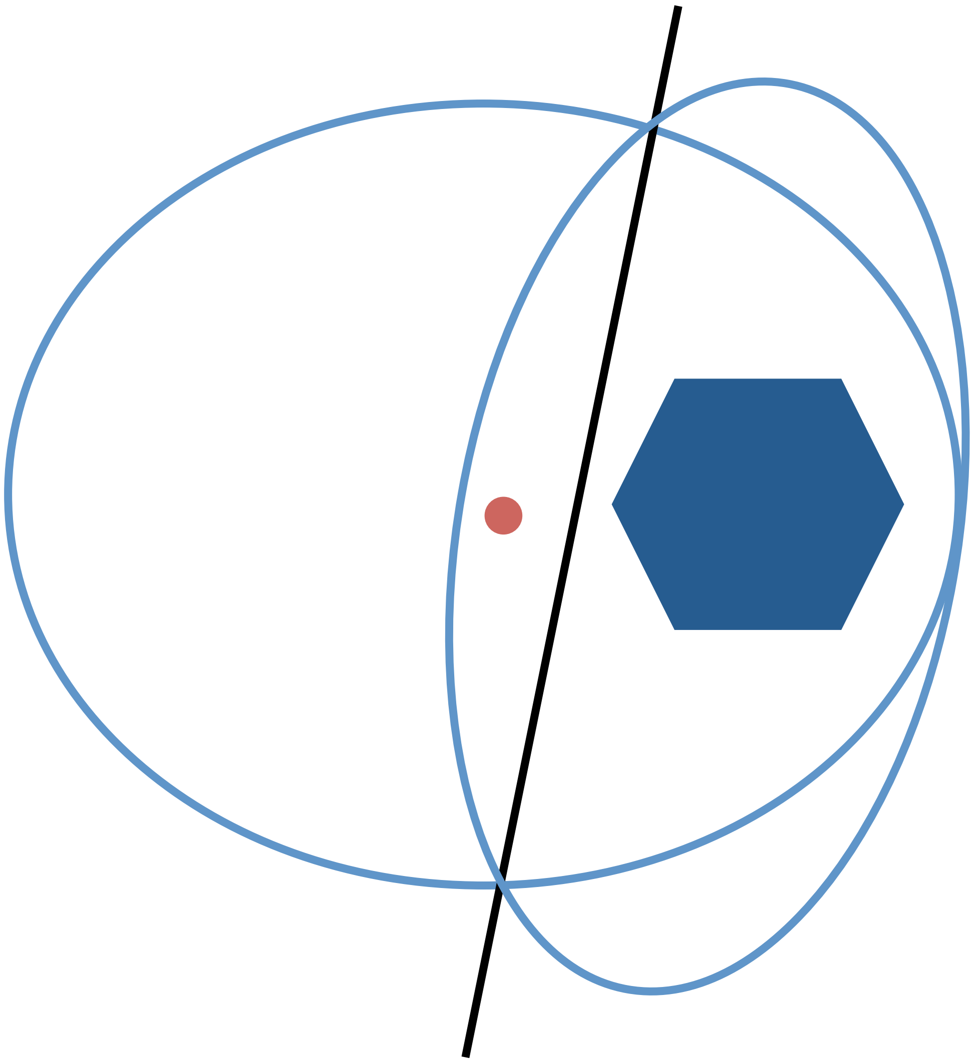
Centroid of ellipsoid

Feasibility via Ellipsoid Method



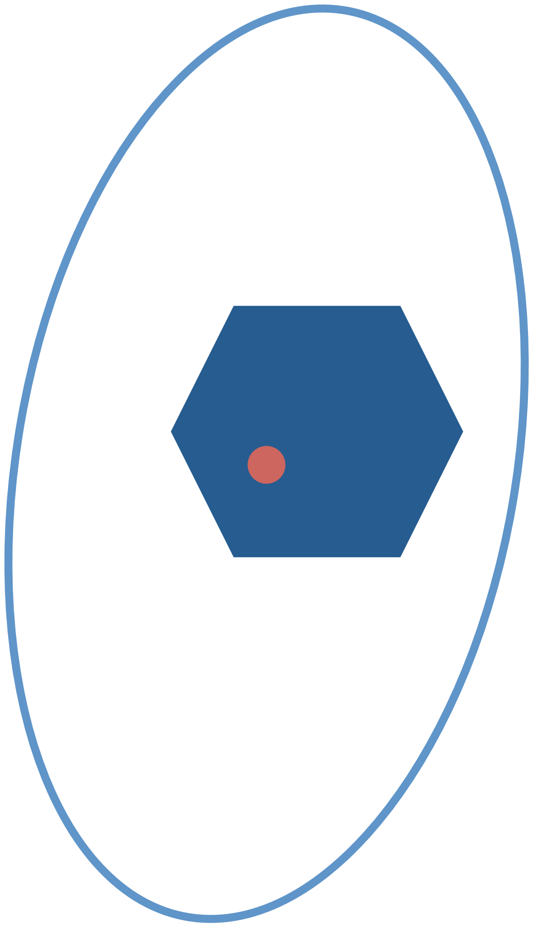
Separating hyperplane for centroid

Feasibility via Ellipsoid Method



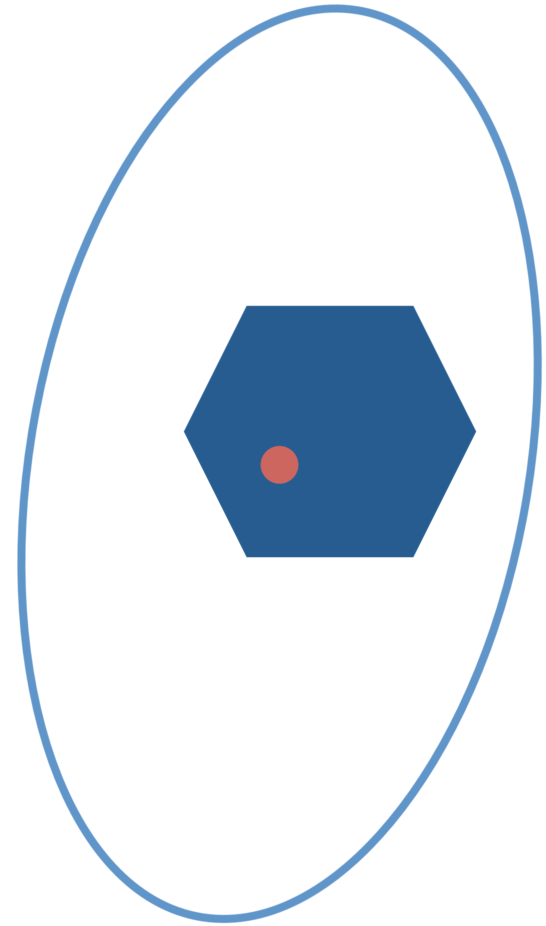
Smallest ellipsoid containing “truncated” ellipsoid

Feasibility via Ellipsoid Method



Centroid of ellipsoid

Feasibility via Ellipsoid Method



Terminate when feasible solution is found

Ellipsoid Method

- Separating hyperplane in polynomial time
  - Check each of the ‘m’ LP constraints in O(n) time
- New ellipsoid in polynomial time
  - Shor (1971), Nemirovsky and Yudin (1972)
- Polynomial iterations (Khachiyan)
  - Volume of ellipsoid reduces exponentially
- Only requires a separation oracle
  - Constraint matrix A can be very large

Useful Fact !!

Questions?