Graph Cuts and Linear Programming

Topic 2.2: Integer Linear Programming

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Outline

- **Integer Linear Programming**
- Duality
- Integer Polyhedron
- Totally Unimodular Matrices

Linear Program

$$\max_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x}$$

$$\text{s.t.} \ \ A\,\mathbf{x} \leq \mathbf{b}$$

Integer Linear Program

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$

$$\text{s.t. } A\,\mathbf{x} \leq \mathbf{b}$$

x is an integer vector

Every element of **x** is an integer

Integer Linear Program

$$\max_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x}$$

$$\text{s.t.} \ A\,\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \in Z^n$$

Every element of **x** is an integer

Example

$\max_{\mathbf{x}} \; x_1 + x_2$

s.t. $\quad x_1 \geq 0$

$x_2 \geq 0$

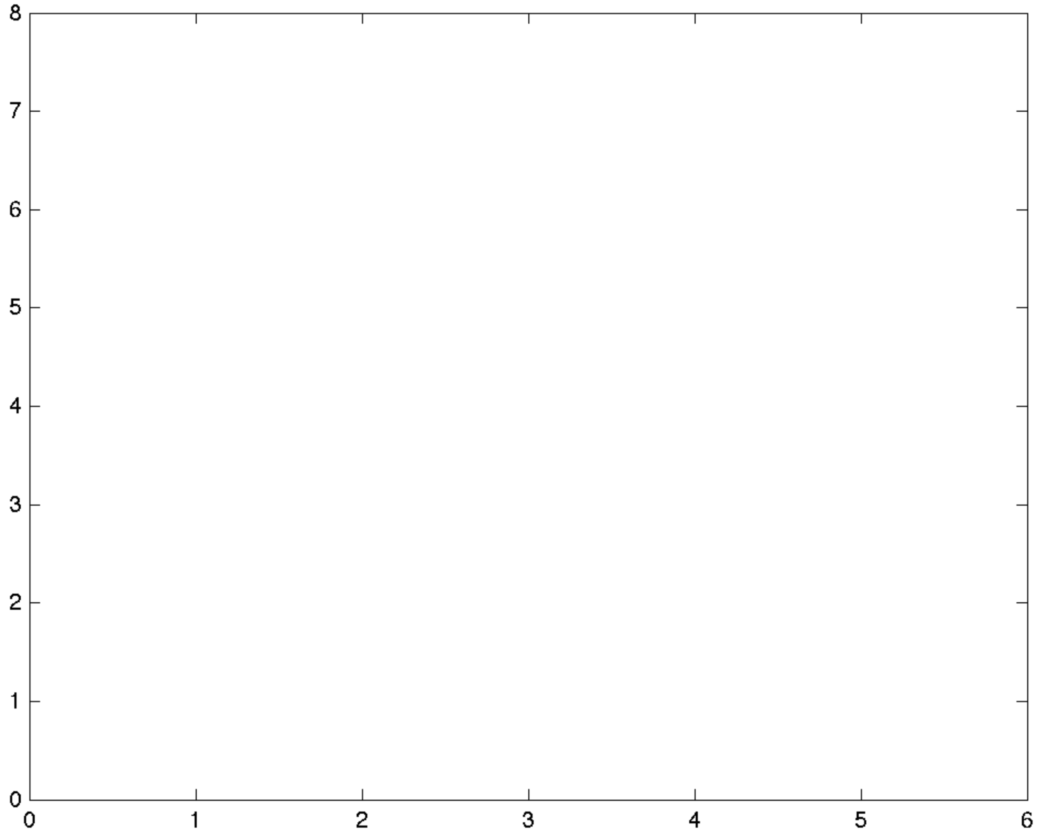
$4x_1 - x_2 \leq 8$

$2x_1 + x_2 \leq 10$

$5x_1 - 2x_2 \geq -2$

$\mathbf{x} \in Z^n$

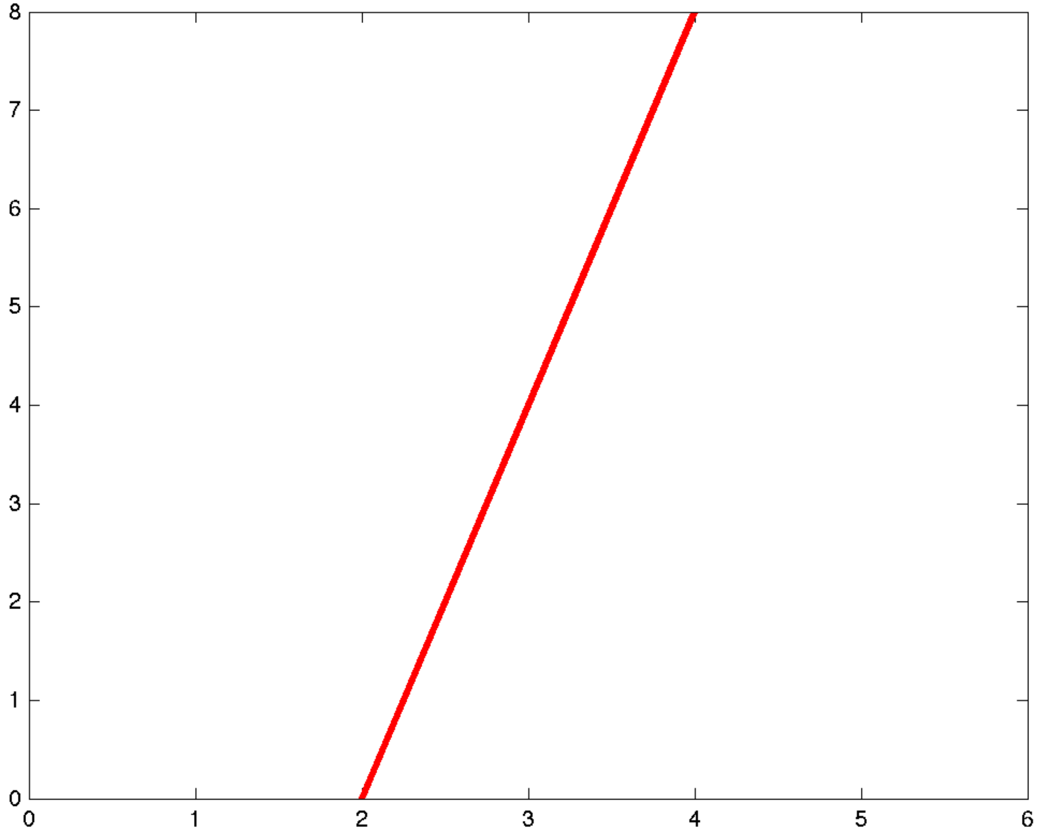
Example



$x_1 \geq 0$

$x_2 \geq 0$

Example



$x_1 \geq 0$

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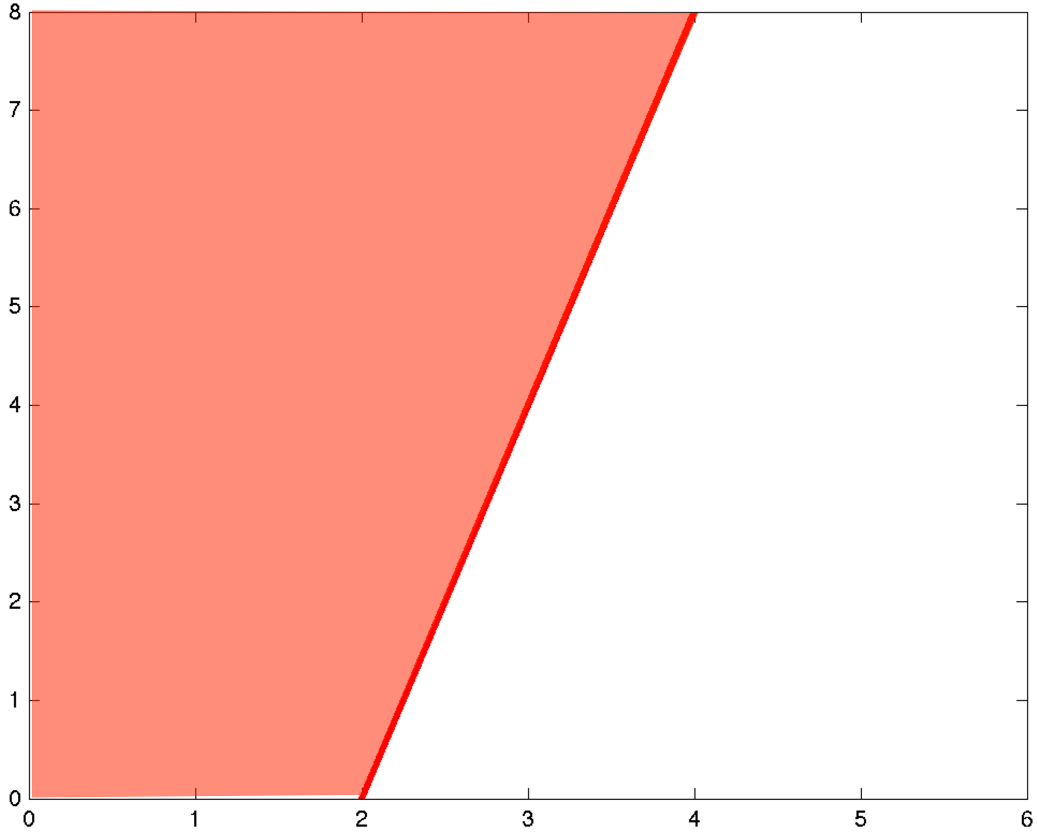
$4x_1 - x_2 = 8$

Example



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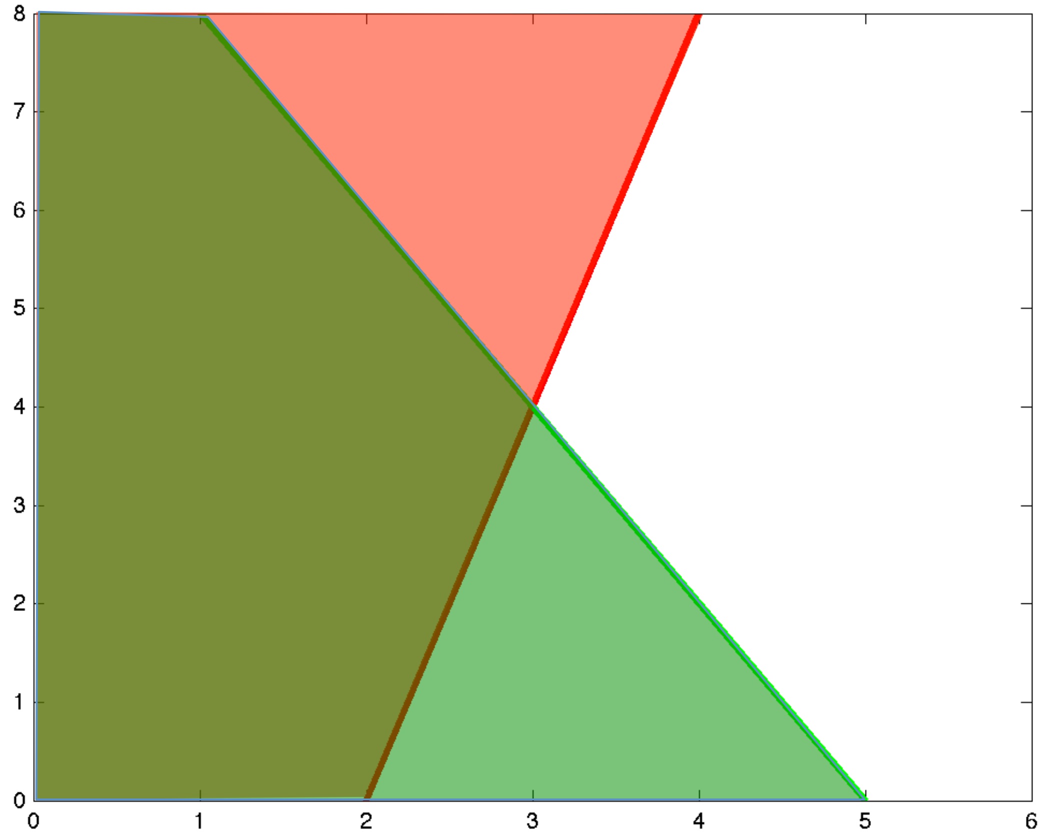

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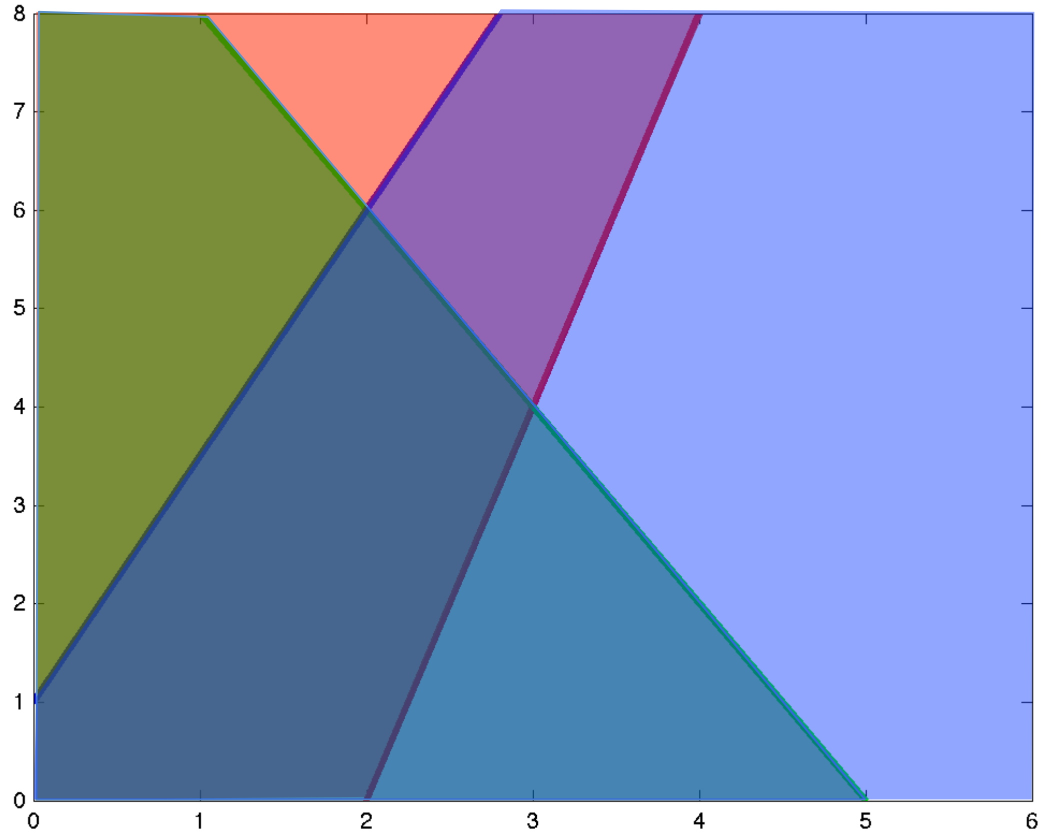

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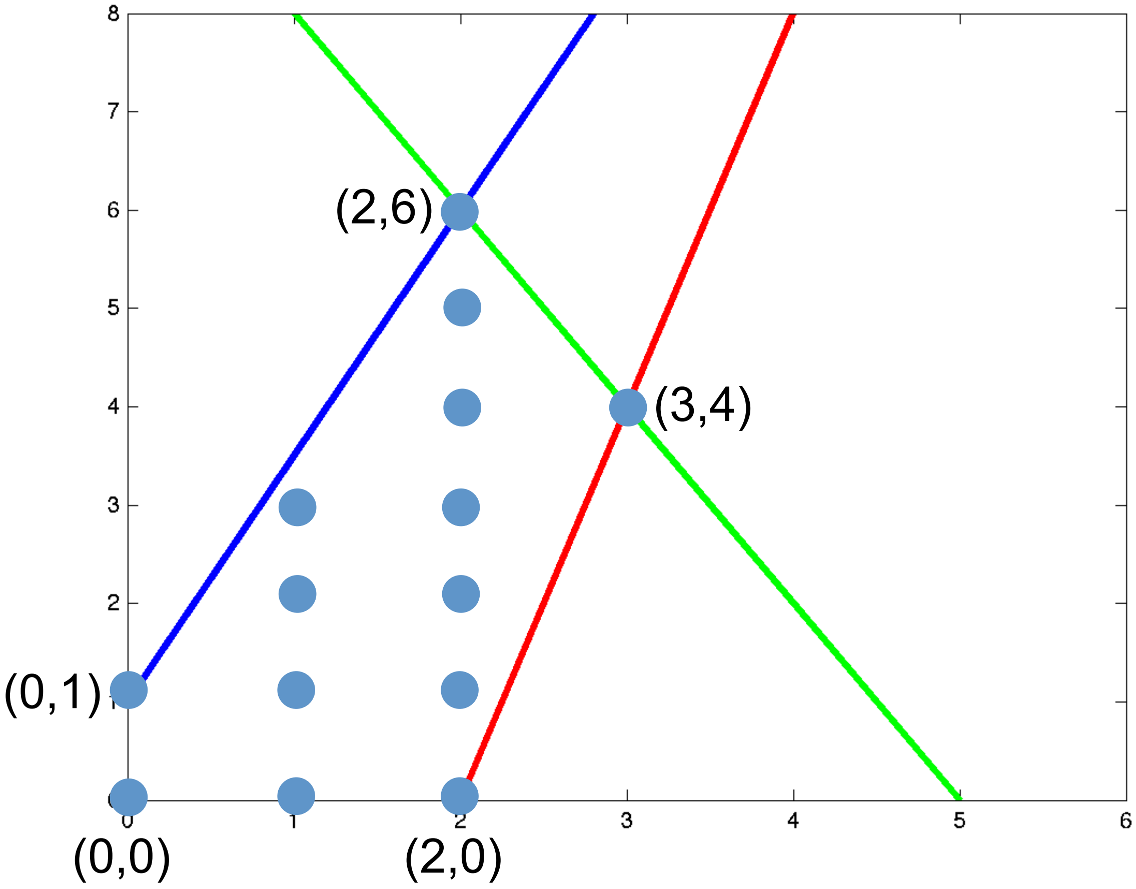

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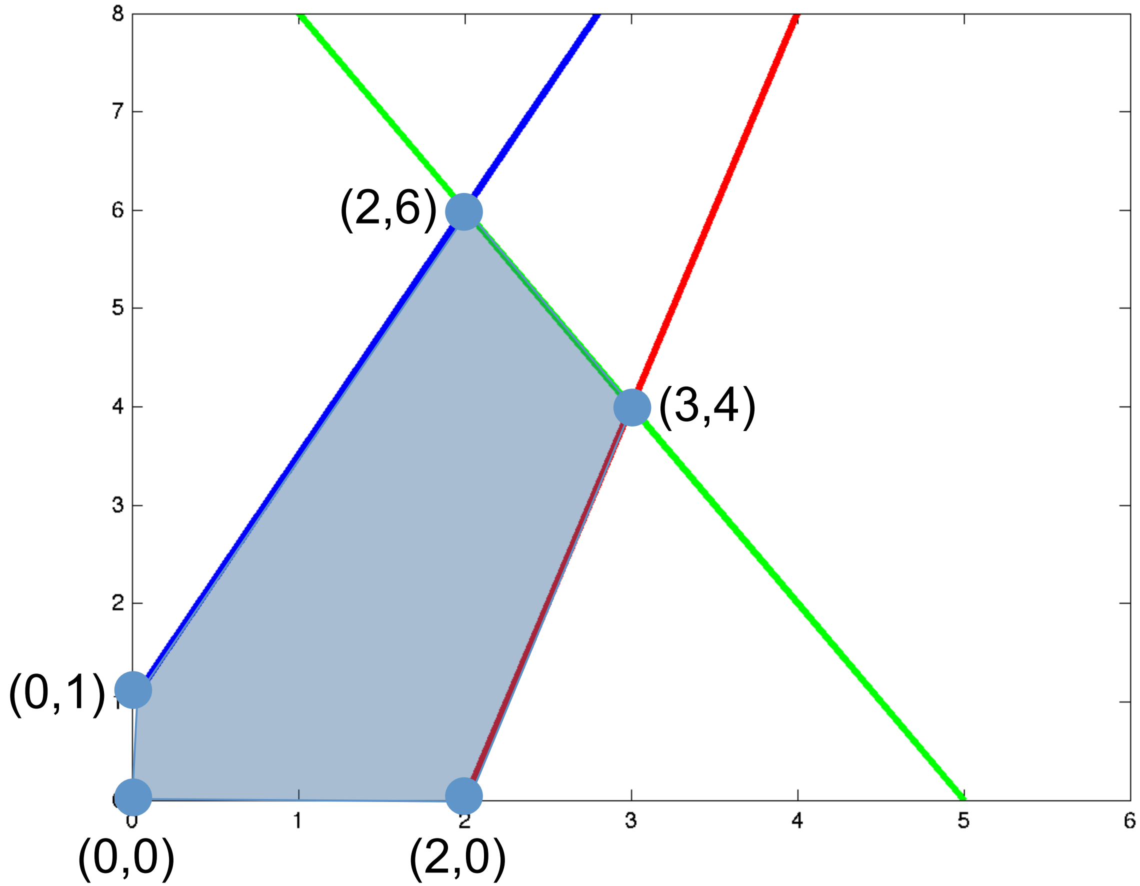
$\mathbf{x} \in Z^n$

$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$

Example



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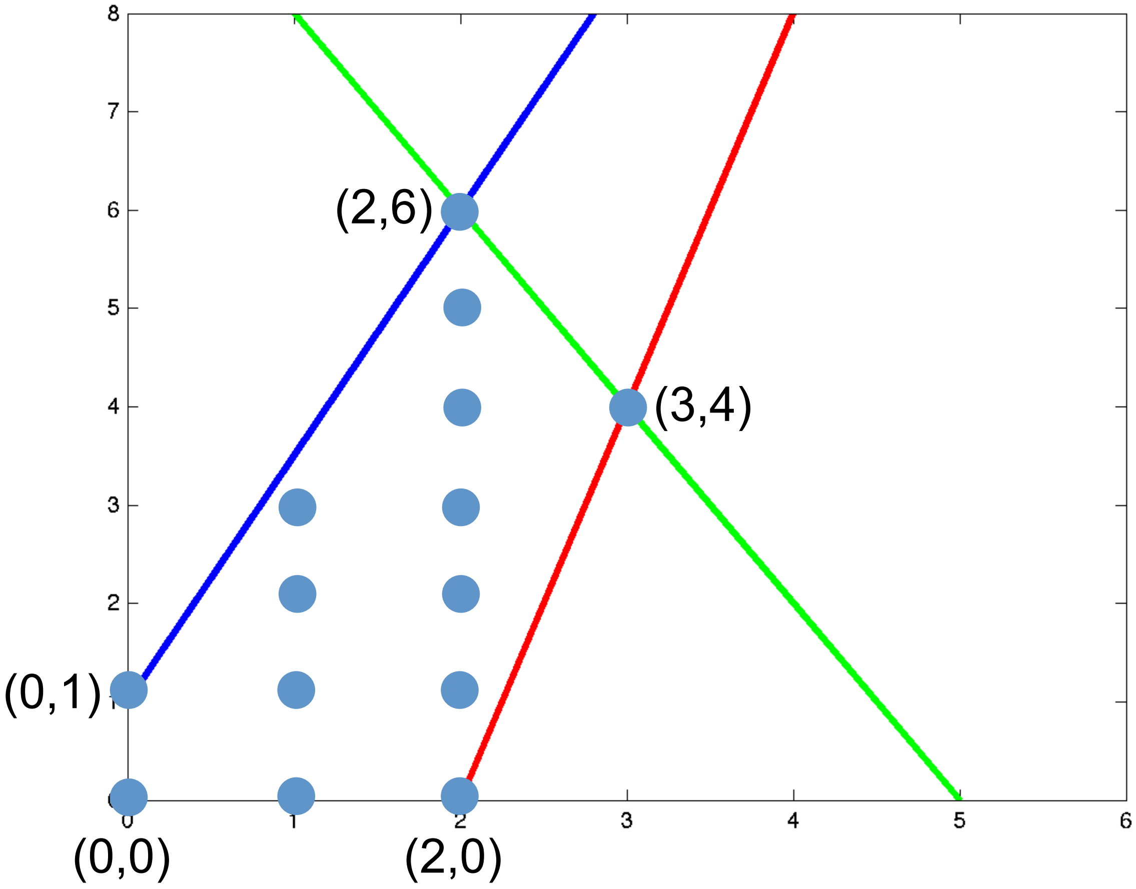
$\cancel{\mathbf{x} \in Z^n}$

$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$

Why? True in general?

Example



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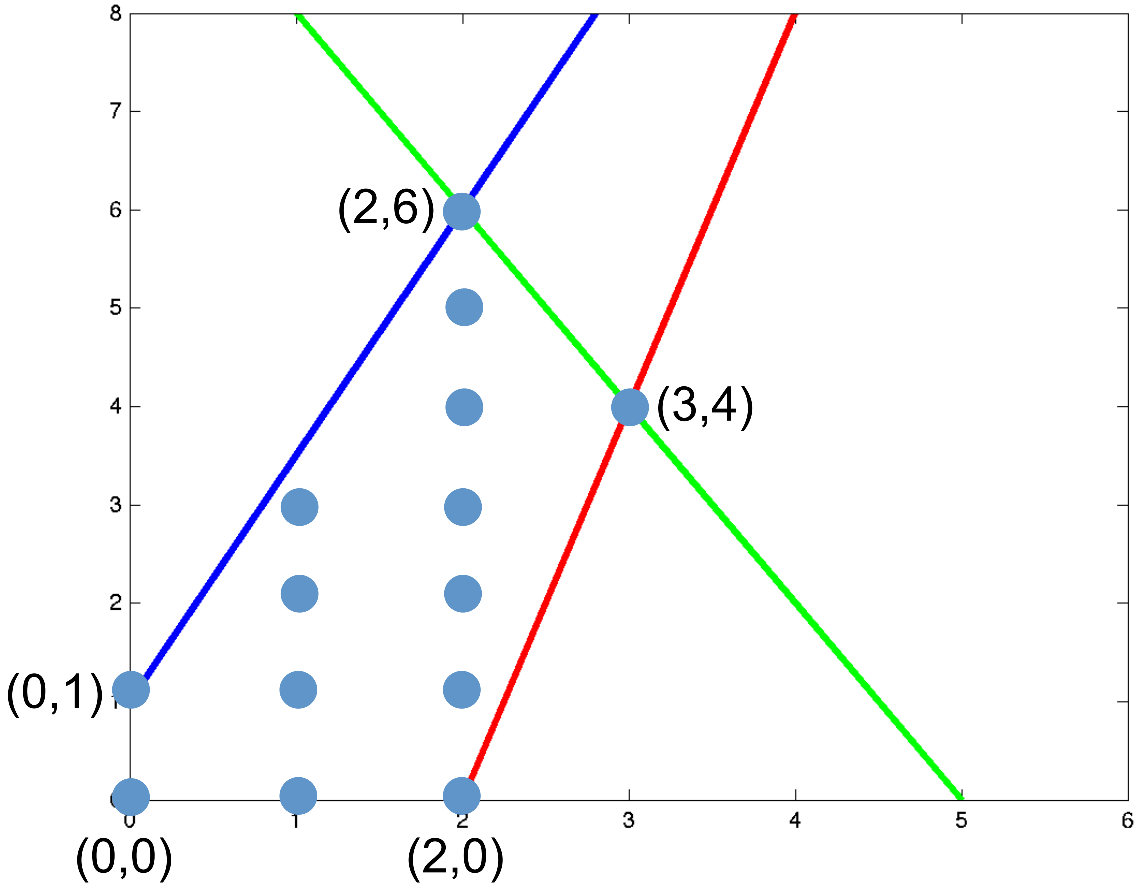

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$\mathbf{x} \in Z^n$

$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$

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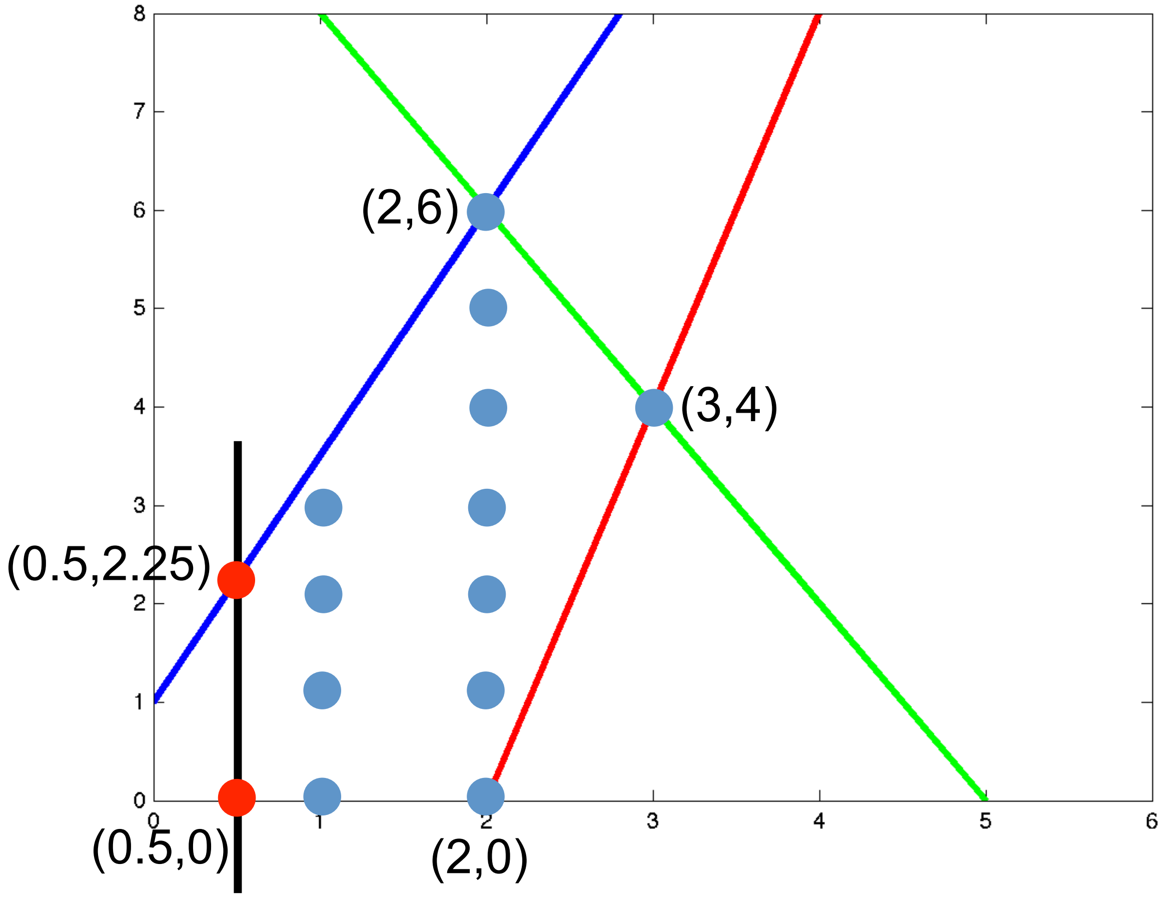

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$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$

Integer Linear Program

ILP are generally NP-hard

Many combinatorial optimization problems

“Easy” ILP = Easy optimization problem

Outline

- Integer Linear Programming
- **Duality**
- Integer Polyhedron
- Totally Unimodular Matrices

Question

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$
$$\text{s.t. } A\,\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \in Z^n$$

$\leq$

$\geq$

$=$

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$
$$\text{s.t. } A\,\mathbf{x} \leq \mathbf{b}$$

Answer

$$\boxed{\begin{aligned} &\max_{\mathbf{x}}\ \mathbf{c}^T\mathbf{x} \\ &\text{s.t.}\ \ A\,\mathbf{x} \le \mathbf{b} \\ &\qquad \mathbf{x} \in Z^n \end{aligned}} \quad \le \quad \boxed{\begin{aligned} &\max_{\mathbf{x}}\ \mathbf{c}^T\mathbf{x} \\ &\text{s.t.}\ \ A\,\mathbf{x} \le \mathbf{b} \end{aligned}}$$

Question

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$
$$\text{s.t. } A\,\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \in Z^n$$

$$\leq$$

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$
$$\text{s.t. } A\,\mathbf{x} \leq \mathbf{b}$$

$$\leq$$
$$\geq$$
$$=$$

$$\min_{\mathbf{y}} \mathbf{b}^T\mathbf{y}$$
$$\text{s.t. } A\,\mathbf{y} = \mathbf{c}$$

Answer

$$\boxed{\begin{aligned} &\max_{\mathbf{x}}\ \mathbf{c}^T\mathbf{x} \\ &\text{s.t. } A\,\mathbf{x} \le \mathbf{b} \\ &\qquad \mathbf{x} \in Z^n \end{aligned}} \quad \le \quad \boxed{\begin{aligned} &\max_{\mathbf{x}}\ \mathbf{c}^T\mathbf{x} \\ &\text{s.t. } A\,\mathbf{x} \le \mathbf{b} \end{aligned}}$$

$$=$$

$$\boxed{\begin{aligned} &\min_{\mathbf{y}}\ \mathbf{b}^T\mathbf{y} \\ &\text{s.t. } A\,\mathbf{y} = \mathbf{c} \end{aligned}}$$

Question

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{s.t. } A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \in Z^n$$

$$\leq$$

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{s.t. } A\mathbf{x} \leq \mathbf{b}$$

$$=$$

$$\min_{\mathbf{y}} \mathbf{b}^T\mathbf{y} \quad \text{s.t. } A\mathbf{y} = \mathbf{c}, \quad \mathbf{y} \in Z^m$$

$$\leq \quad \geq \quad =$$

$$\min_{\mathbf{y}} \mathbf{b}^T\mathbf{y} \quad \text{s.t. } A\mathbf{y} = \mathbf{c}$$

Answer

$$\boxed{\begin{aligned} &\max_x \mathbf{c}^T\mathbf{x} \\ &\text{s.t. } A\,\mathbf{x} \le \mathbf{b} \\ &\qquad \mathbf{x} \in Z^n \end{aligned}} \quad \le \quad \boxed{\begin{aligned} &\max_x \mathbf{c}^T\mathbf{x} \\ &\text{s.t. } A\,\mathbf{x} \le \mathbf{b} \end{aligned}}$$

$$=$$

$$\boxed{\begin{aligned} &\min_y \mathbf{b}^T\mathbf{y} \\ &\text{s.t. } A\,\mathbf{y} = \mathbf{c} \\ &\qquad \mathbf{y} \in Z^m \end{aligned}} \quad \ge \quad \boxed{\begin{aligned} &\min_y \mathbf{b}^T\mathbf{y} \\ &\text{s.t. } A\,\mathbf{y} = \mathbf{c} \end{aligned}}$$

Duality Relationship

$$\max_x \mathbf{c}^T\mathbf{x} \quad \text{s.t. } A\,\mathbf{x} \le \mathbf{b}, \quad \mathbf{x} \in Z^n$$

$$\le$$

$$\max_x \mathbf{c}^T\mathbf{x} \quad \text{s.t. } A\,\mathbf{x} \le \mathbf{b}$$

$$\le$$

$$=$$

$$\min_y \mathbf{b}^T\mathbf{y} \quad \text{s.t. } A\,\mathbf{y} = \mathbf{c}, \quad \mathbf{y} \in Z^m$$

$$\ge$$

$$\min_y \mathbf{b}^T\mathbf{y} \quad \text{s.t. } A\,\mathbf{y} = \mathbf{c}$$

Duality Relationship

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{s.t. } A\,\mathbf{x} \le \mathbf{b},\ \mathbf{x} \in Z^n$$

$\le$

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{s.t. } A\,\mathbf{x} \le \mathbf{b}$$

$\le$

Strict inequality for some problems

$$\min_{\mathbf{y}} \mathbf{b}^T\mathbf{y} \quad \text{s.t. } A\,\mathbf{y} = \mathbf{c},\ \mathbf{y} \in Z^m$$

$\ge$

$=$

$$\min_{\mathbf{y}} \mathbf{b}^T\mathbf{y} \quad \text{s.t. } A\,\mathbf{y} = \mathbf{c}$$

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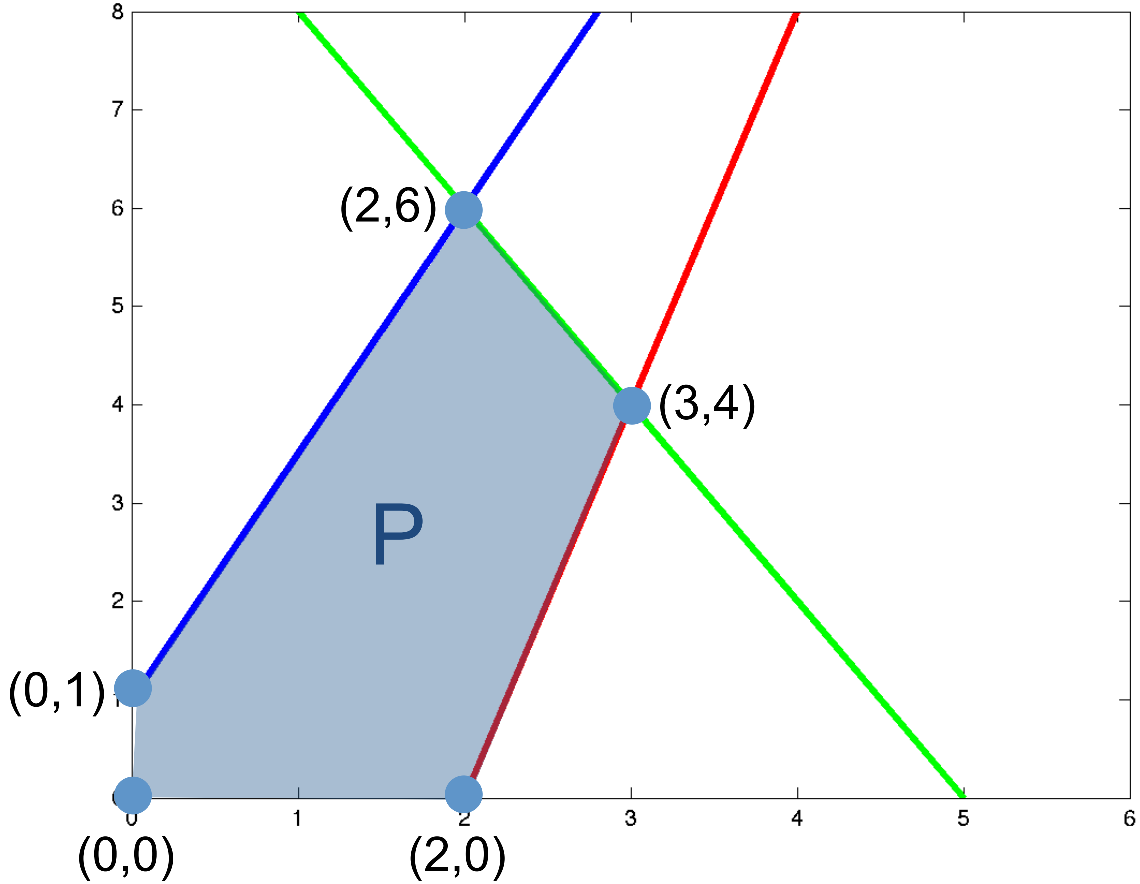
- Duality
- **Integer Polyhedron**
- Totally Unimodular Matrices

Integer Polyhedron



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$4x_1 - x_2 \leq 8$

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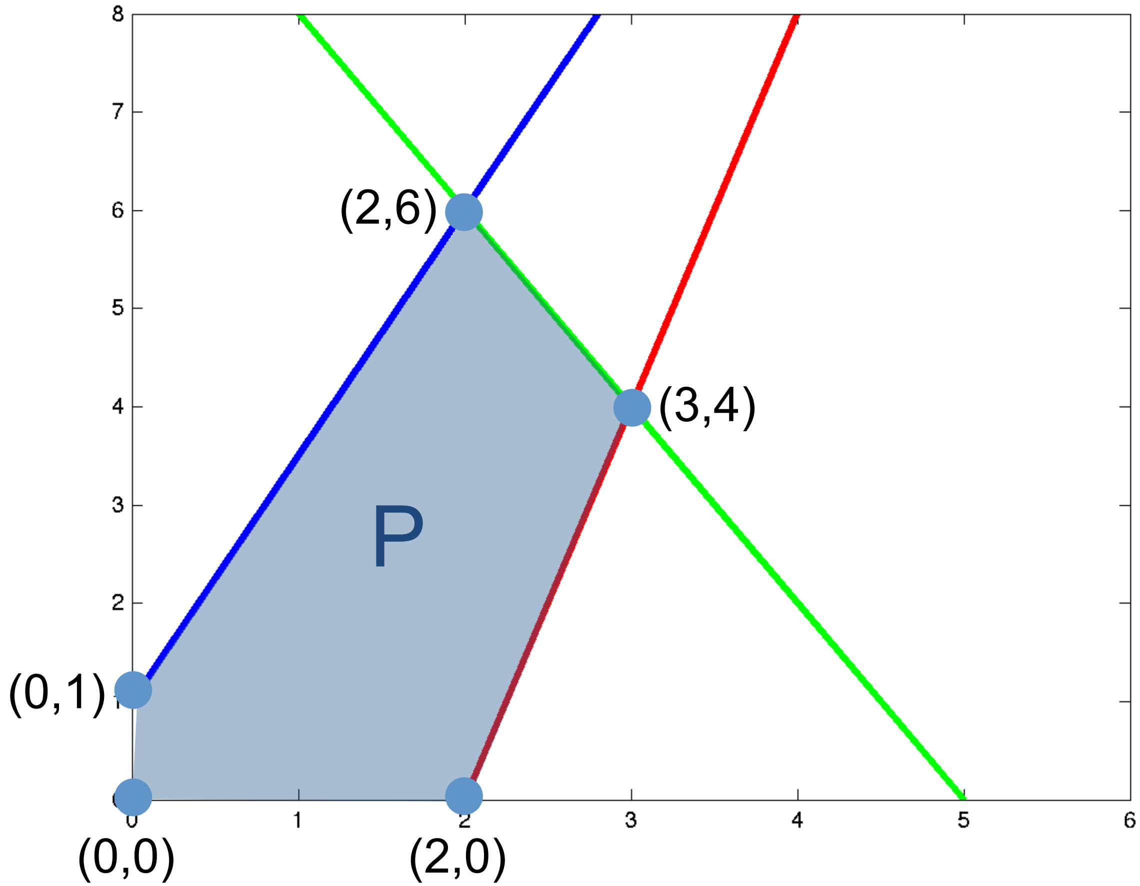
All the vertices of P are integer vectors

Integer polytope is a bounded integer polyhedron

Integer Polyhedron



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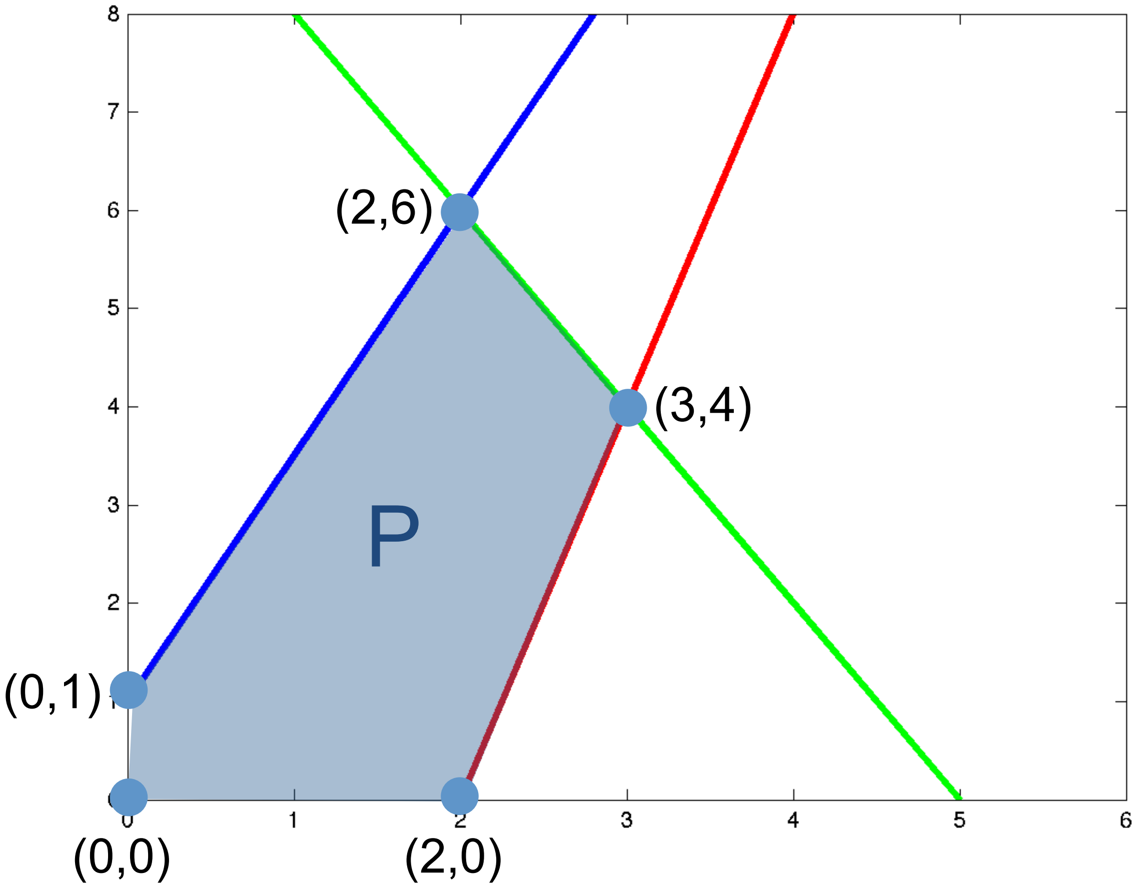
Integer polyhedron need not be bounded

Integer Polyhedron



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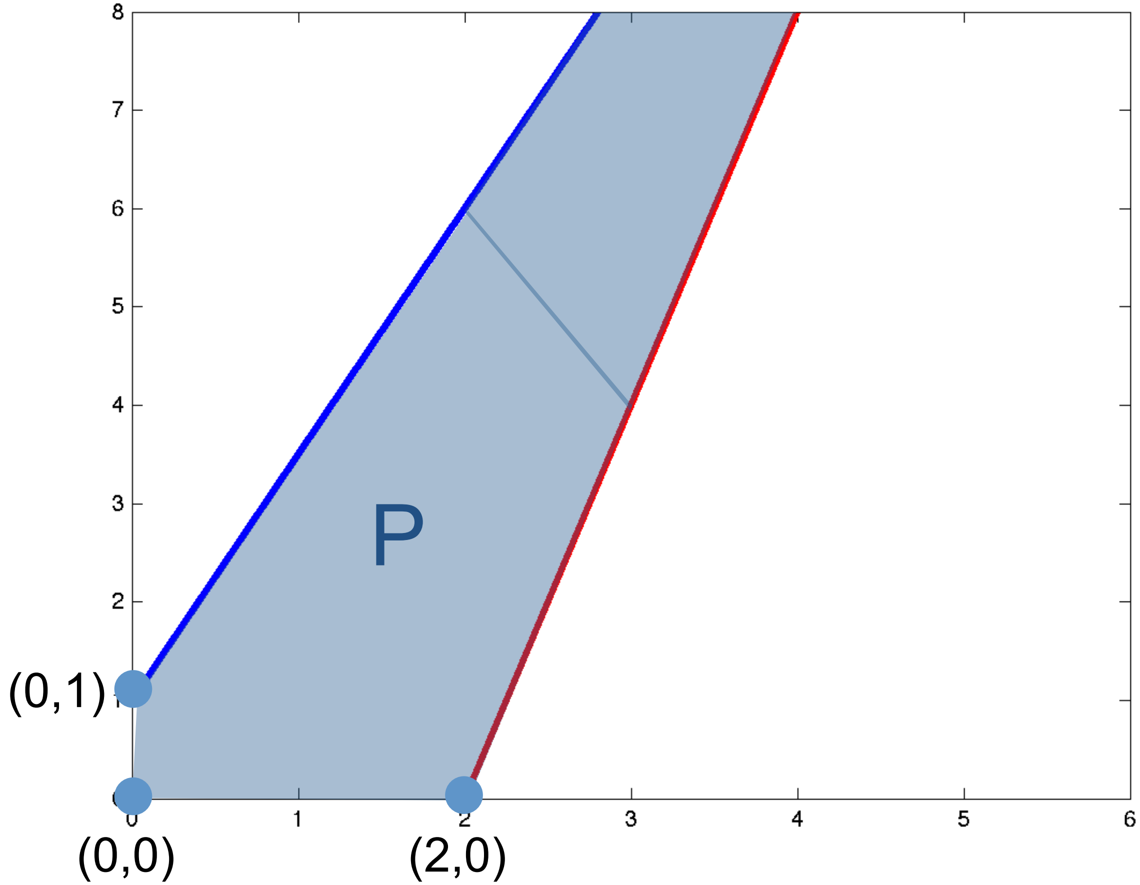

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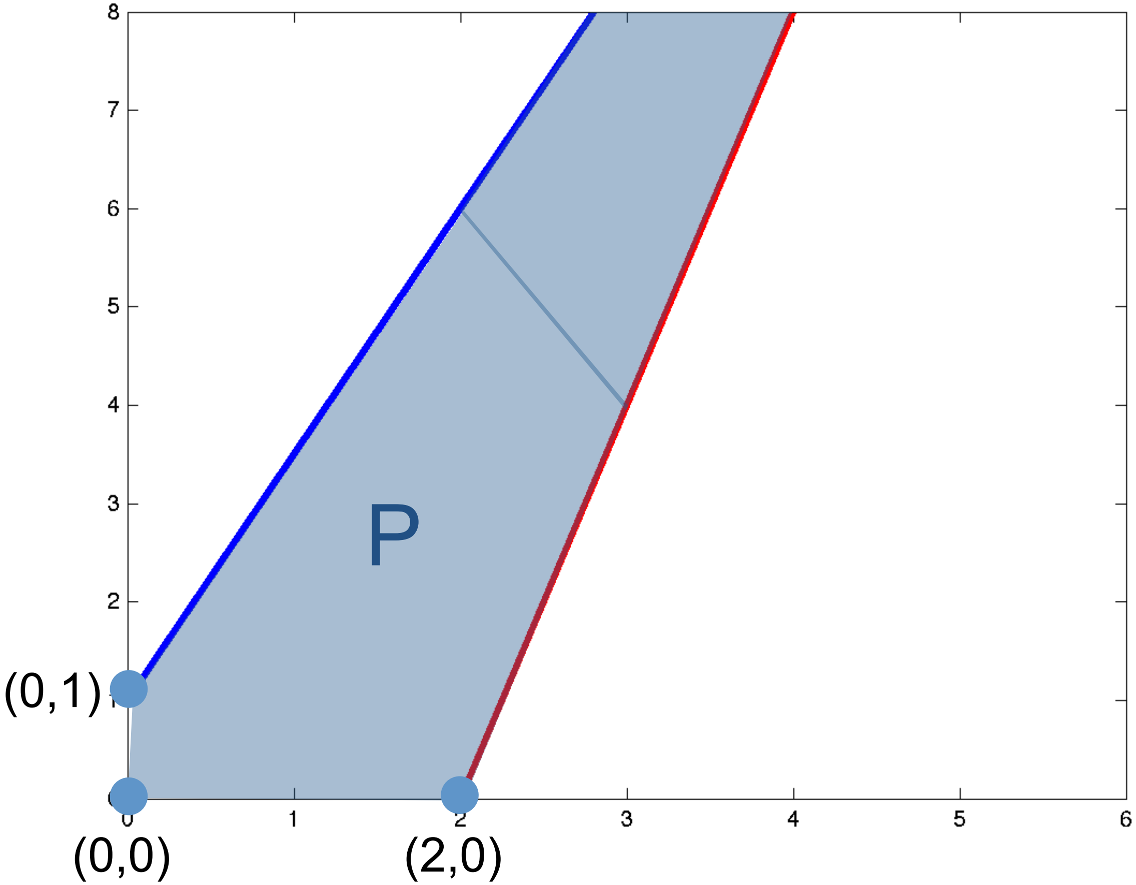

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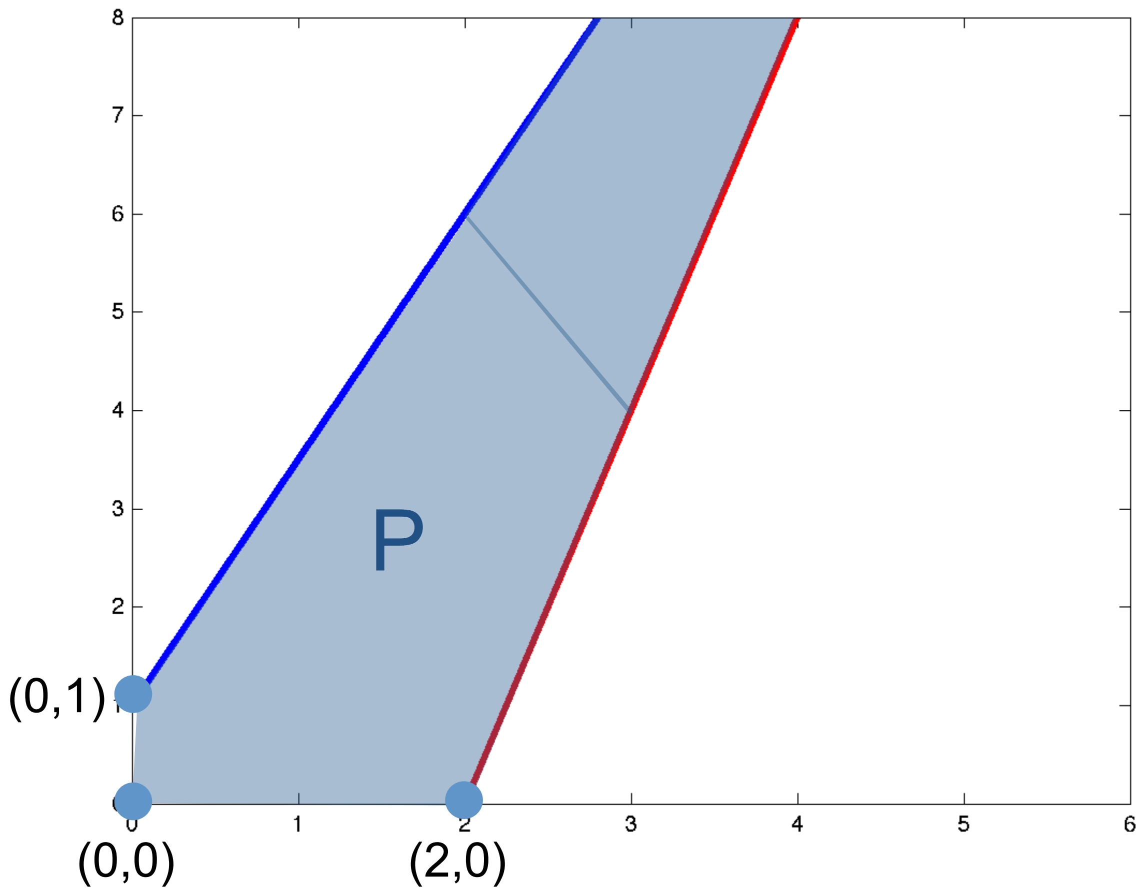
Are all polyhedra integer polyhedra? NO

Question



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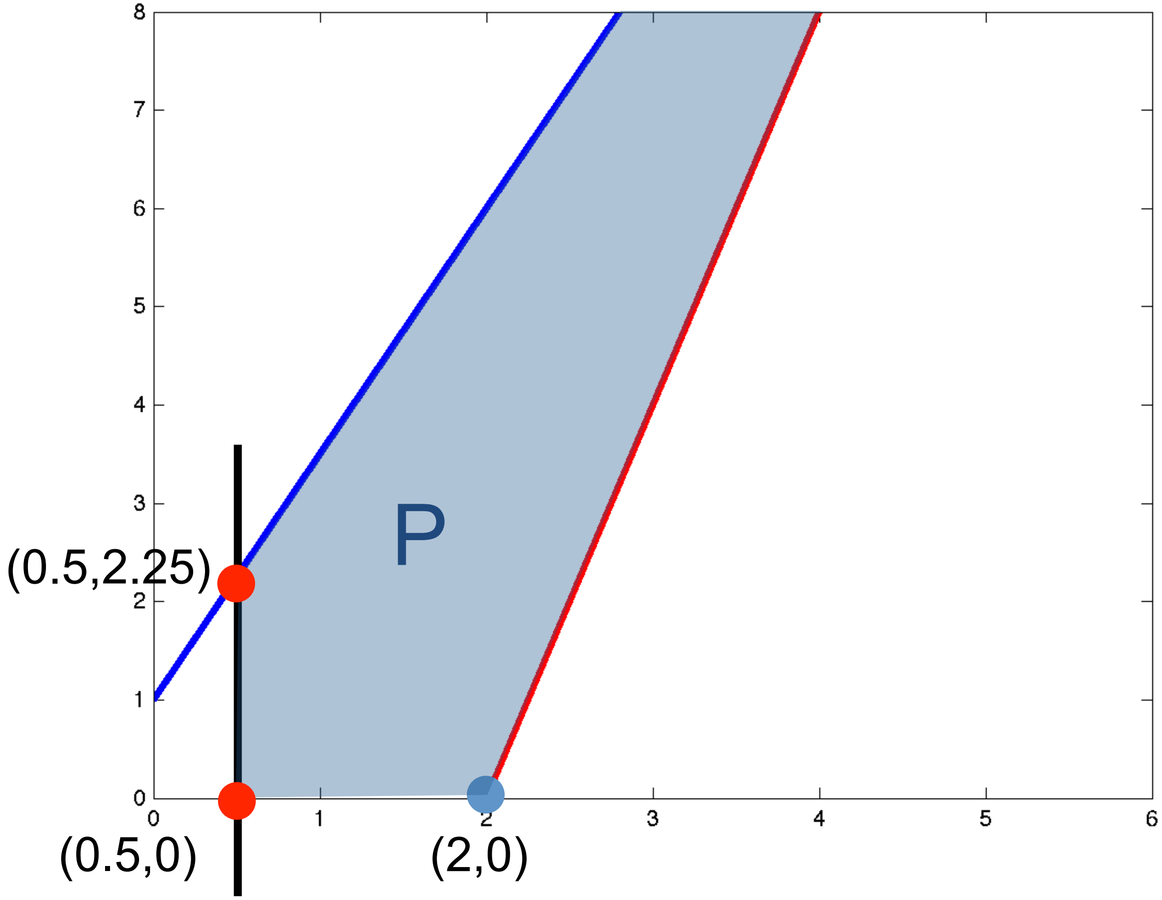

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Integer Polyhedron

Integer polyhedra are very useful

ILP over Integer polyhedron is easy

Drop the integrality constraints, solve LP

But how can we identify integer polyhedra?

Outline

- Integer Linear Programming
- Duality
- Integer Polyhedron
- **Totally Unimodular Matrices**

Totally Unimodular Matrix

A is a TUM

For all square submatrix A’ of A

det(A’) is 0, +1 or -1

Question

$$\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$$

Is this a TUM?

NO

All elements must be 0, +1 or -1

Question

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Is this a TUM?

NO

Determinant of the matrix = -2

Question

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

Is this a TUM?

YES.

Question

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Is this a TUM?

YES.

Property

A is a TUM

b is an integer vector, $\mathbf{b} \in Z^n$

Polyhedron $P = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$

P is an integer polyhedron **Proof?**

Proof Sketch

Let **v** be a vertex of P.

A' is a full-rank square submatrix of A

b' is the corresponding subvector of **b**

A'**v** = **b**'   **Why?**

Proof Sketch

Let **v** be a vertex of P. **v** is integer

A' is a full-rank square submatrix of A

b' is the corresponding subvector of **b**

Integer +1 or -1

A'**v** = **b**' **v** = adj(A')**b**'/det(A')

Totally Unimodular Matrix

How can we identify if A is TUM?

$O((m+n)^3)$ algorithm. K. Truemper, 1990

We don't want to identify easy problem instances

We want to identify easy problems

Totally Unimodular Matrix

Some types of matrices are TUM

Easy to prove that they are TUM

Corresponding problems are provably “easy”

We will see that min-cut is one such problem

Questions?