Neural Network Verification

M. Pawan Kumar

Personal webpage http://mpawankumar.info

Tutorial webpage http://mpawankumar.info/tutorials/vmcai2019/index.html

Tutorial Outline

- Neural Networks
- Formulation of Verification
- Unsound Methods
- Incomplete Methods
- Complete Methods

Neural Network Verification

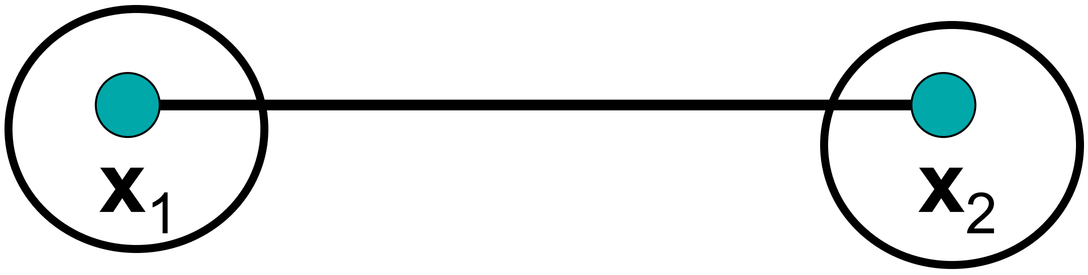
Part 0: *Preliminaries*

Outline

- Convex Sets
- Convex Functions
- Convex Program

Convex Set

Line Segment

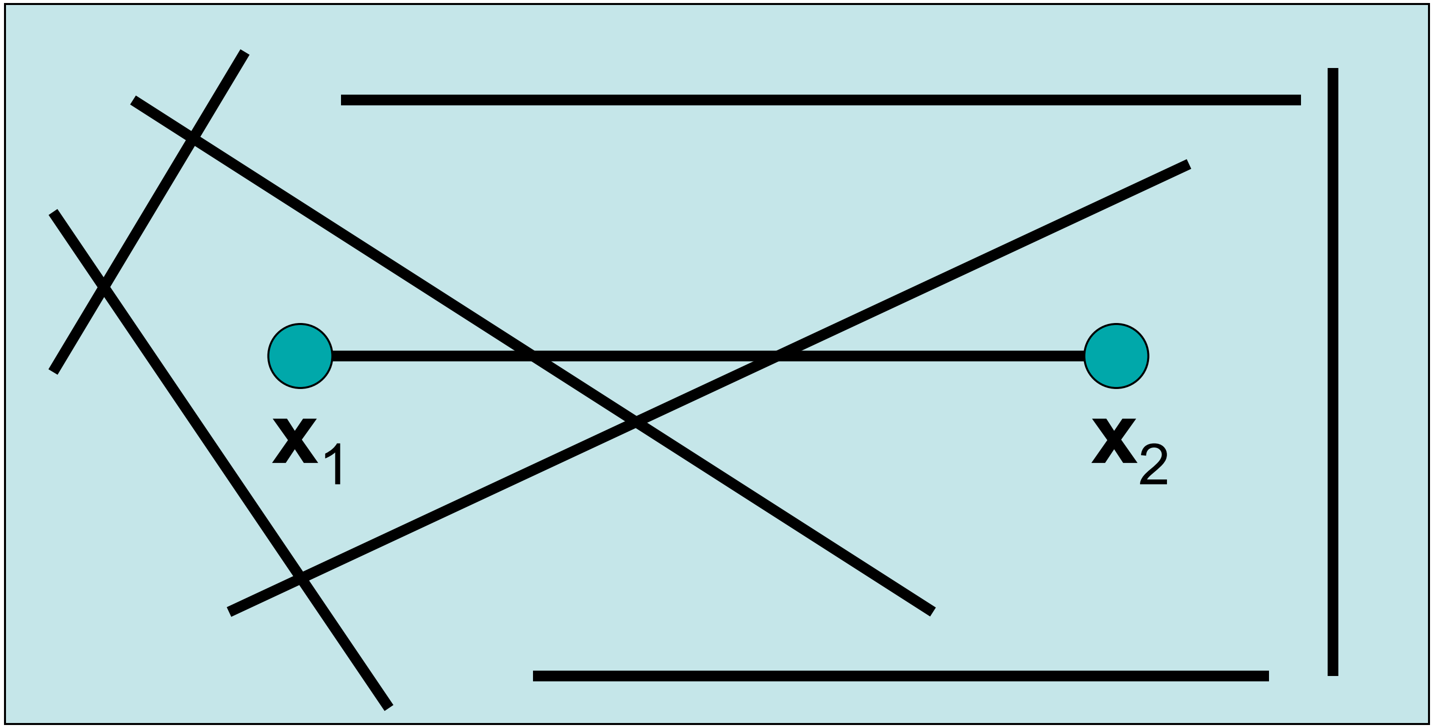


$$\lambda\, \mathbf{x}_1 + (1 - \lambda)\, \mathbf{x}_2$$

$$\lambda \in [0,1]$$

Endpoints

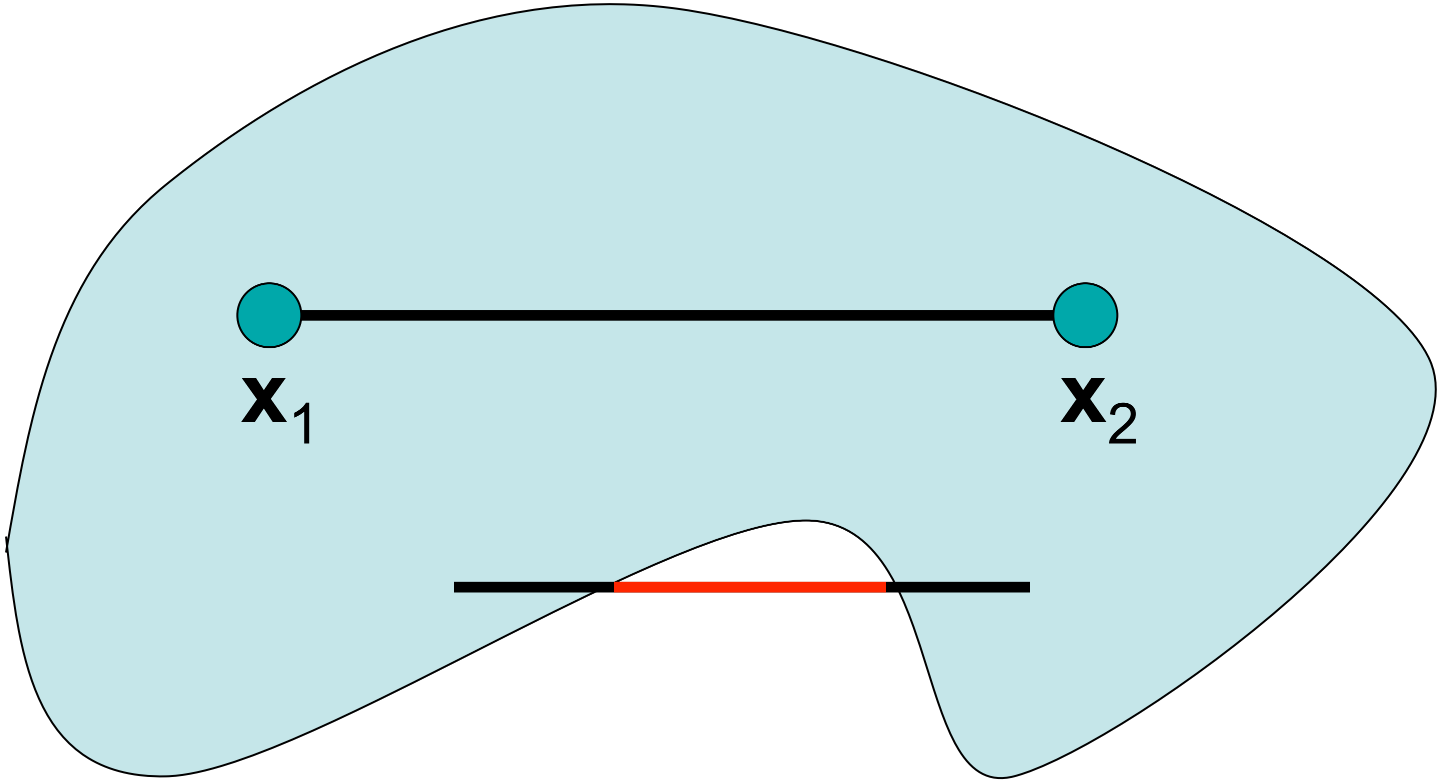
Convex Set



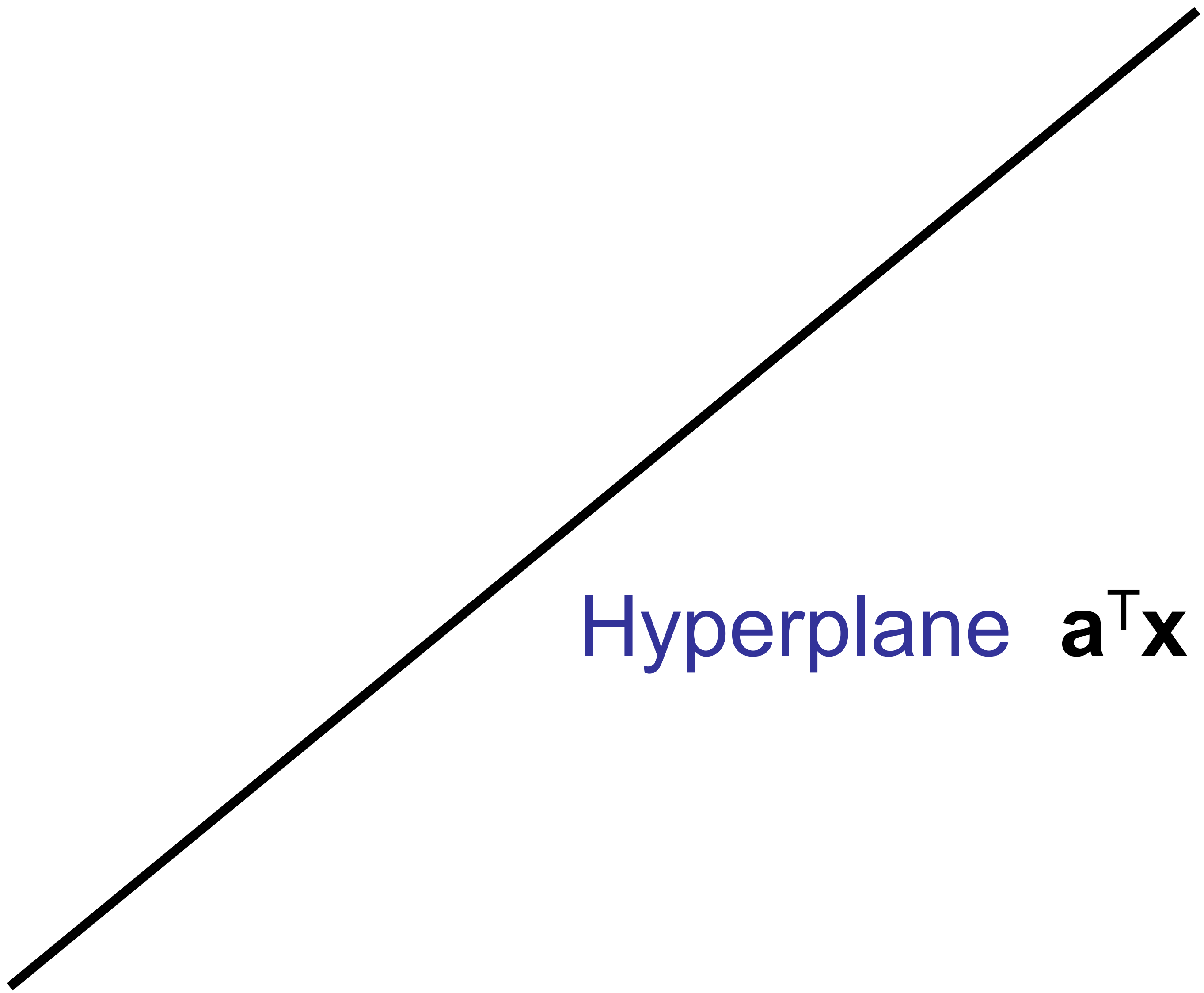
All points on the line segment lie within the set

For all line segments with endpoints in the set

Non-Convex Set

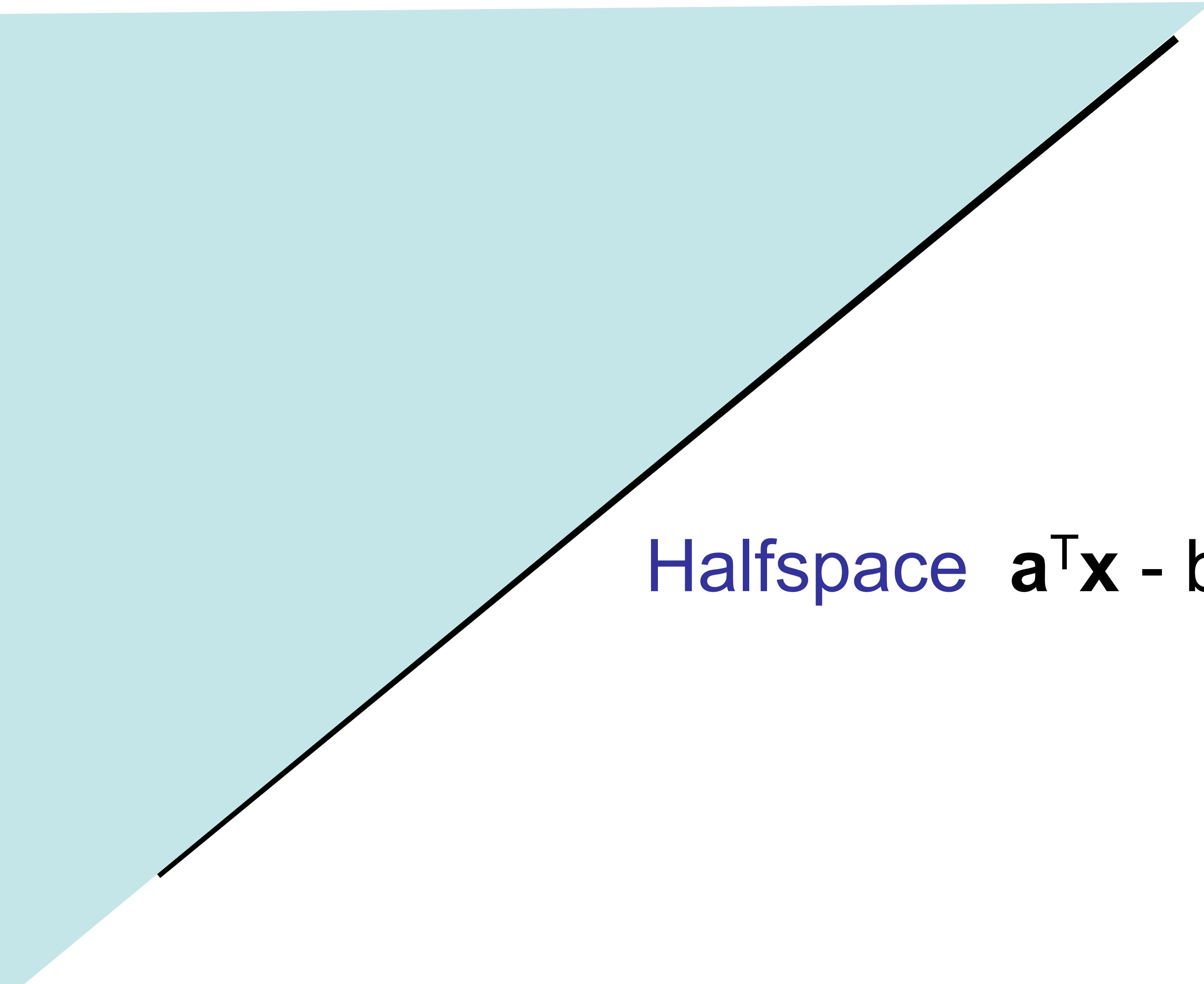


Examples of Convex Sets



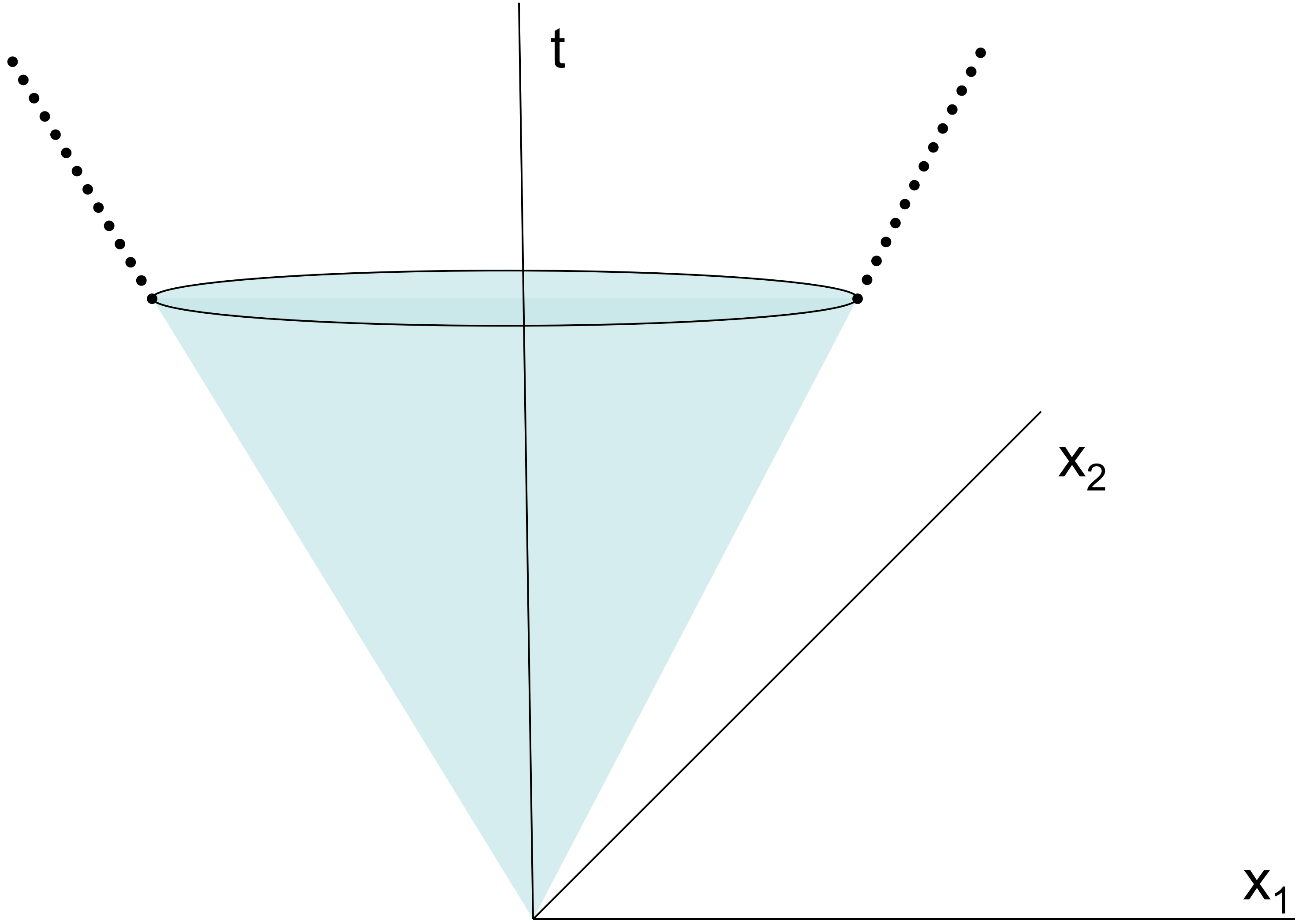
Hyperplane $\mathbf{a}^T\mathbf{x} - b = 0$

Examples of Convex Sets



Halfspace $\mathbf{a}^T\mathbf{x} - b \leq 0$

Examples of Convex Sets



Second-order Cone $||\mathbf{x}|| \leq t$

Examples of Convex Sets

Semidefinite Matrices $\{X \mid X \succeq \mathbf{0}\}$

$\mathbf{a}^T X \mathbf{a} \geq \mathbf{0}$, for all $\mathbf{a} \in R^n$

All eigenvalues of X are non-negative

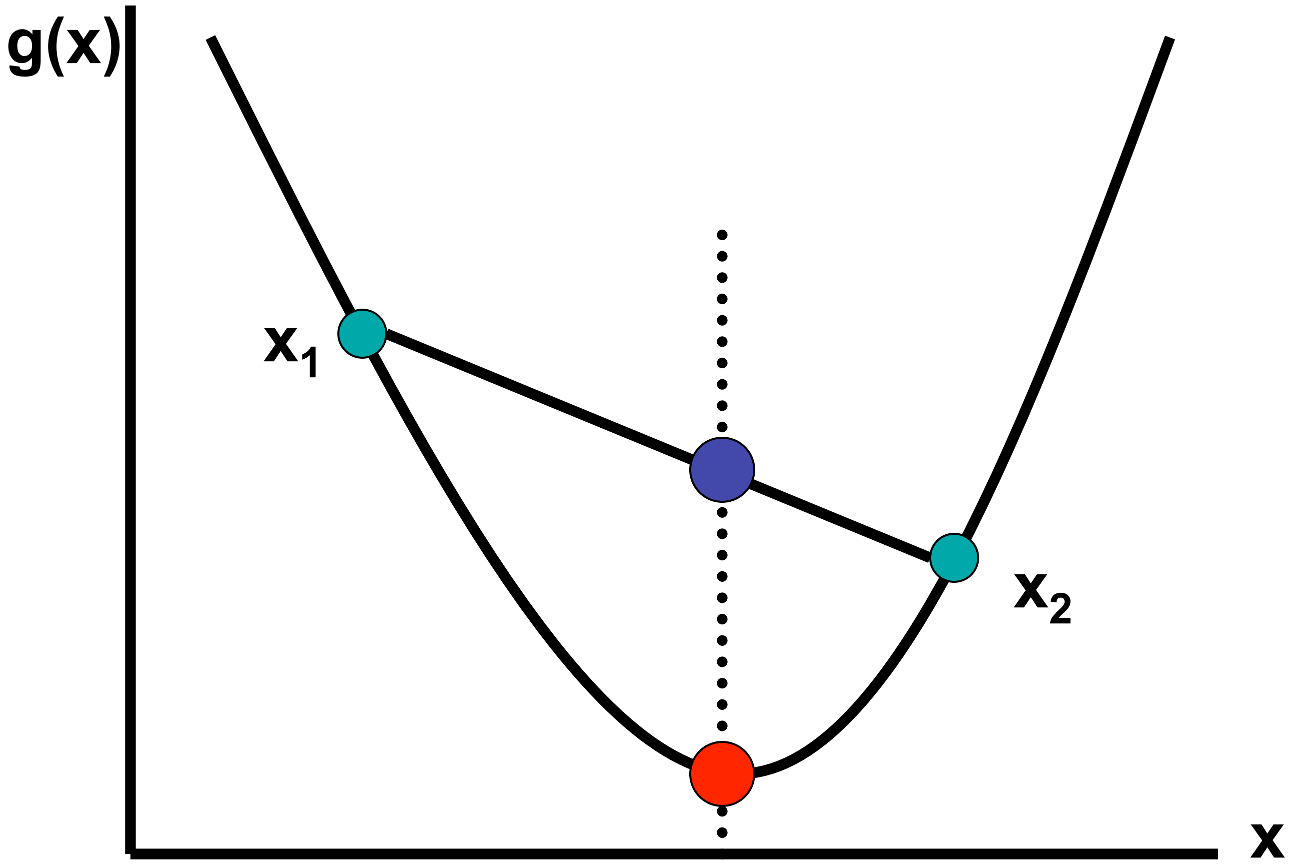
$\mathbf{a}^T X_1 \mathbf{a} \geq \mathbf{0}$ $\mathbf{a}^T X_2 \mathbf{a} \geq \mathbf{0}$

$\mathbf{a}^T(\lambda X_1 + (1-\lambda) X_2)\mathbf{a} \geq \mathbf{0}$

Outline

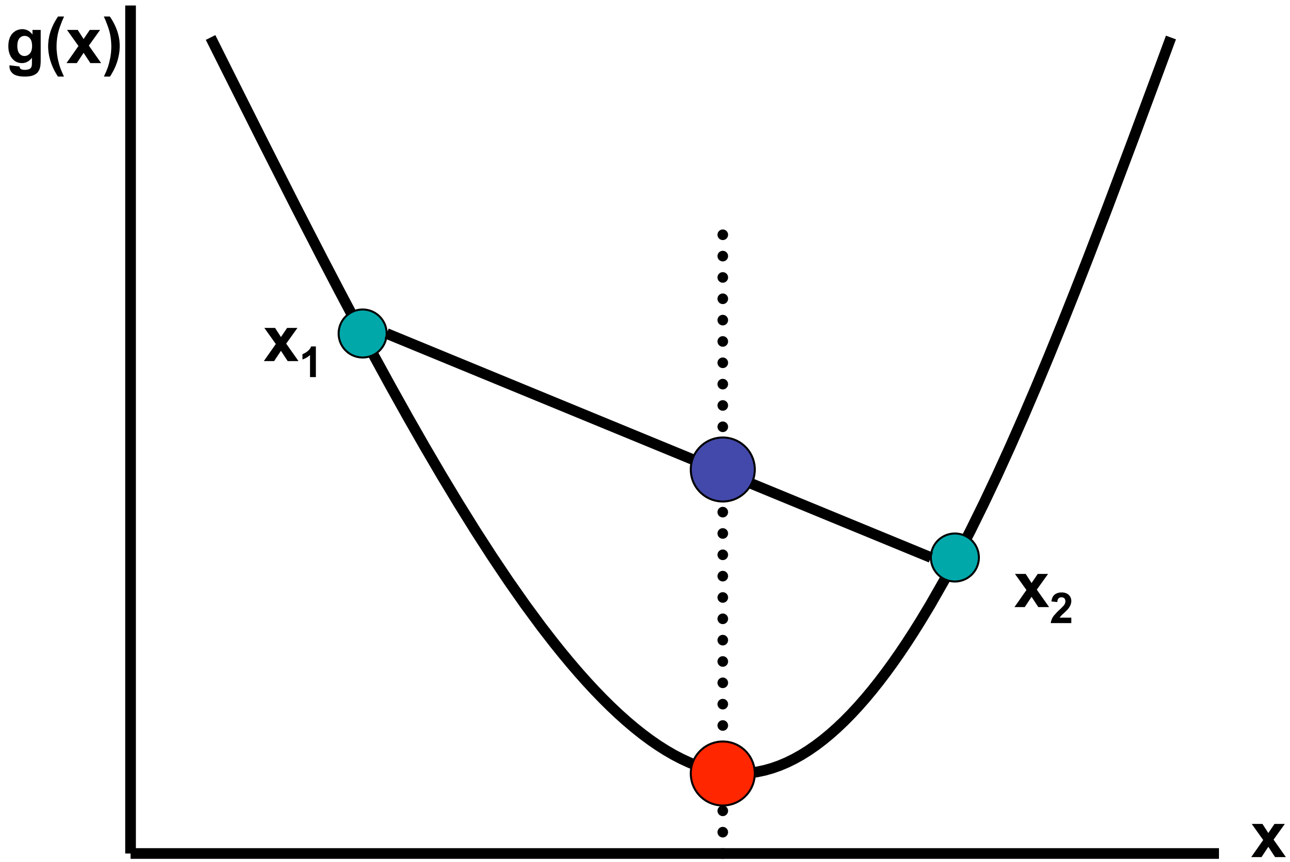
- Convex Sets
- **Convex Functions**
- Convex Program

Convex Function



Blue point always lies above red point

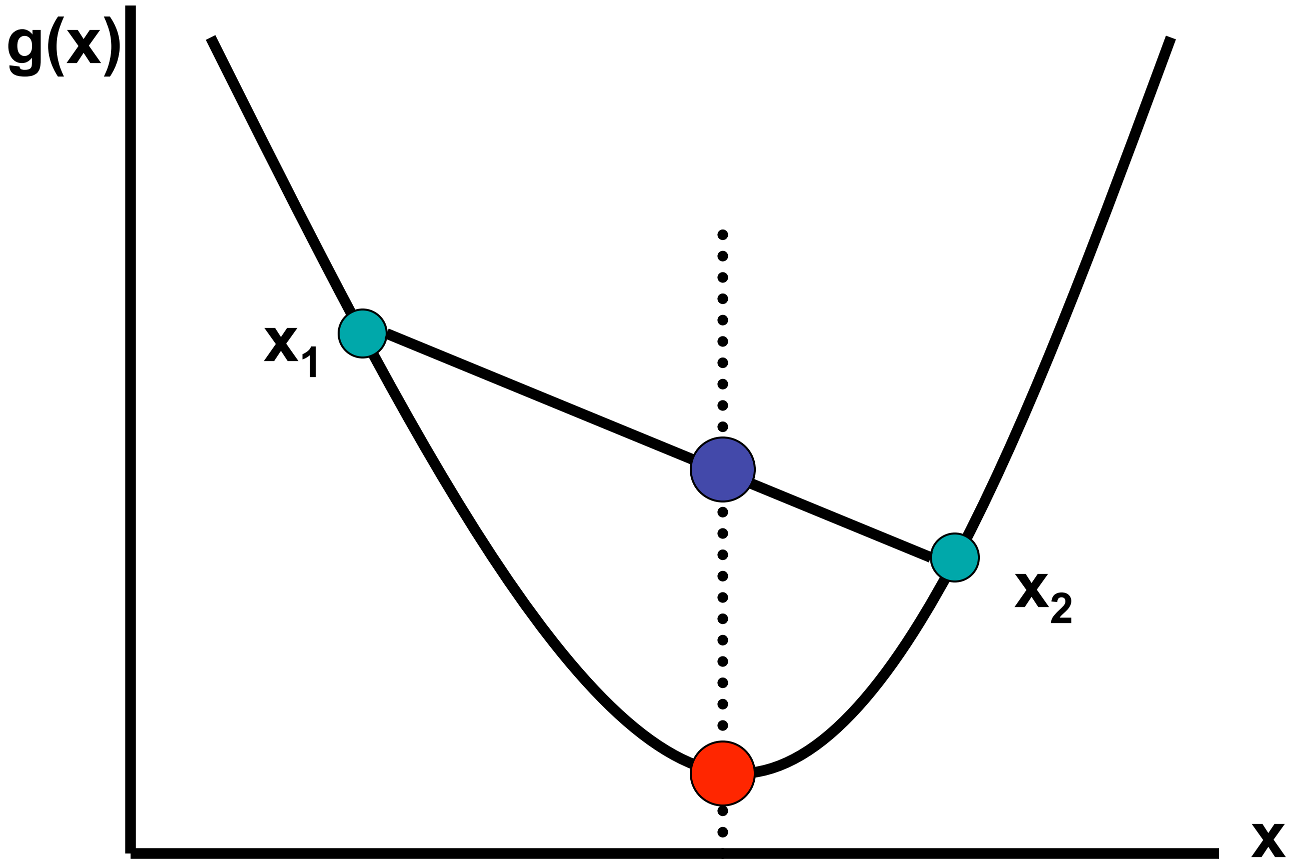
Convex Function



$$g(c\,\mathbf{x}_1 + (1 - c)\,\mathbf{x}_2) \le c\,g(\mathbf{x}_1) + (1 - c)\,g(\mathbf{x}_2)$$

Domain of g(.) has to be convex

Convex Function



$$g(\,c\,\mathbf{x}_1 + (1 - c)\,\mathbf{x}_2\,) \leq c\,g(\mathbf{x}_1) + (1 - c)\,g(\mathbf{x}_2)$$

-g(.) is concave

Examples of Convex Functions

Linear function $\mathbf{a}^T\mathbf{x}$

p-Norm functions $(x_1^p + x_2^p + x_n^p)^{1/p}$, $p \geq 1$

Quadratic functions $\mathbf{x}^T \mathbf{Q} \mathbf{x}$

$$\mathbf{Q} \succeq 0$$

Outline

- Convex Sets
- Convex Functions
- **Convex Program**

Convex Program

Minimize a convex function

$$\min_{\mathbf{x}} f(\mathbf{x})$$ Objective function

$$\text{s.t. } g_i(\mathbf{x}) \leq 0$$ Constraints

$$i = 1, 2, \ldots. n$$

Over a convex feasible region

Local Minima are Global Minima

Unconstrained optimization $\min_{\mathbf{x}} f(\mathbf{x})$

Let $\mathbf{x}^*$ be a local minimum

For all $\mathbf{y}$, $\exists\, \alpha$ small enough such that

$f(\mathbf{x}^*) \leq f((1-\alpha)\mathbf{x}^*+\alpha\mathbf{y})$ $\quad$ $\mathbf{x}^*$ is local minimum

$\leq (1-\alpha)\, f(\mathbf{x}^*)+\alpha\, f(\mathbf{y})$ $\quad$ f is convex

Therefore, $f(\mathbf{x}^*) \leq f(\mathbf{y})$, for all $\mathbf{y}$

Local Minima are Global Minima

Constrained optimization

$$\min_{\mathbf{x}} f(\mathbf{x})$$

$$\text{s.t. } g_i(\mathbf{x}) \leq 0$$
$$i = 1, 2, \ldots. n$$

Indicator function $I_i(\mathbf{x}) = \begin{cases} 0, \text{ if } g_i(\mathbf{x}) \leq 0 \\ \infty, \text{ if } g_i(\mathbf{x}) > 0 \end{cases}$

Local Minima are Global Minima

$$\min_{\mathbf{x}} f(\mathbf{x}) + \Sigma_i \, l_i(\mathbf{x})$$

Unconstrained convex optimization

Apply previous argument

Many convex programs have efficient solvers

Questions?