Neural Network Verification

Part 5: *Complete Methods*

Neural Network Verification

Neural network f

Scalar output $z = f(\mathbf{x})$

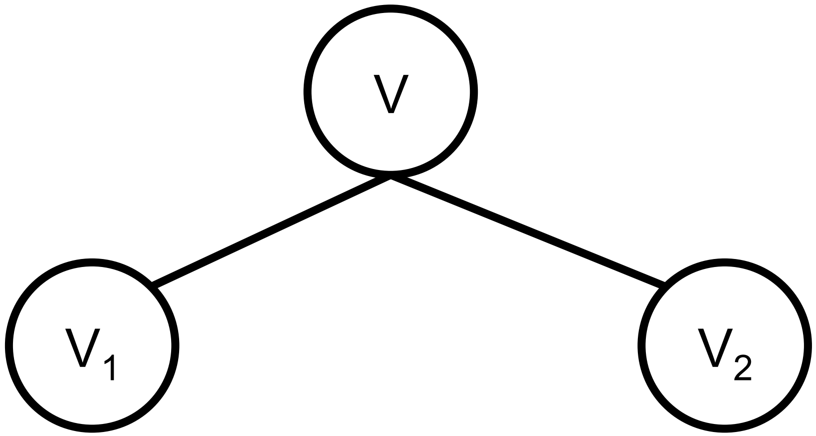
E.g. in binary classification, $z = s(y^*;\mathbf{x}) - s(y;\mathbf{x})$ for $y \neq y^*$

Property: $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in X$

Complete methods try to disprove the property

Branch-and-Bound

Find **v** ∈ V such that h(**v**) ≤ 0

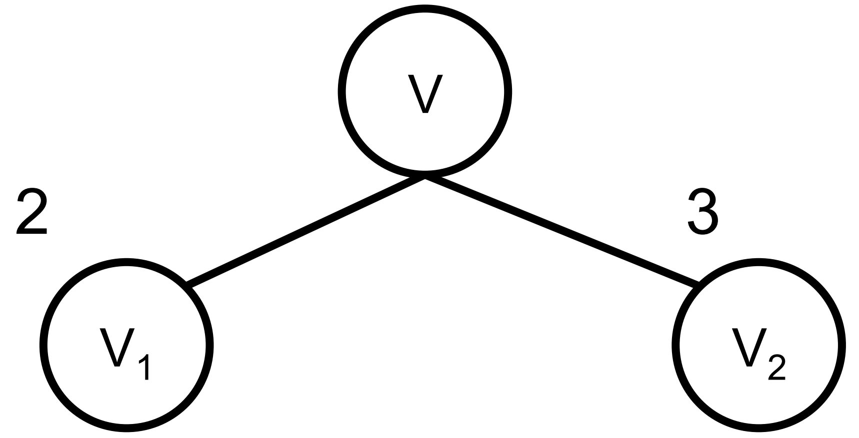


BRANCH: Split the feasible set

2 or more usually disjoint subsets

Branch-and-Bound

Find **v** ∈ V such that h(**v**) ≤ 0

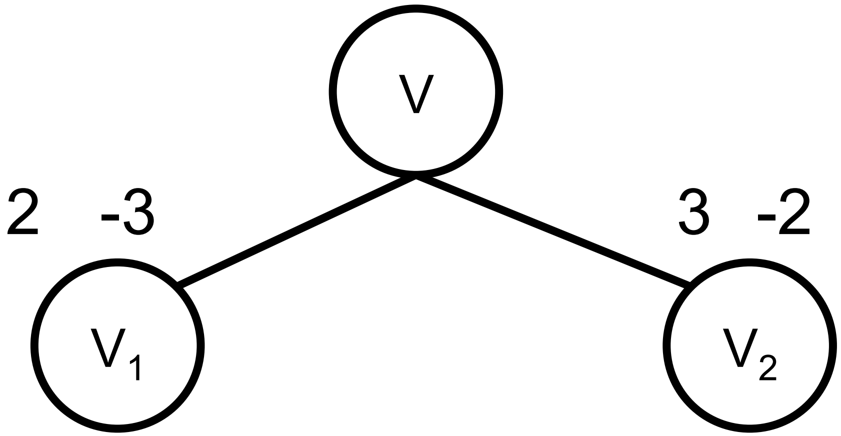


BOUND: Compute upper bounds for each branch

h(**v**) for any feasible **v** (unsound methods)

Branch-and-Bound

Find **v** ∈ V such that h(**v**) ≤ 0

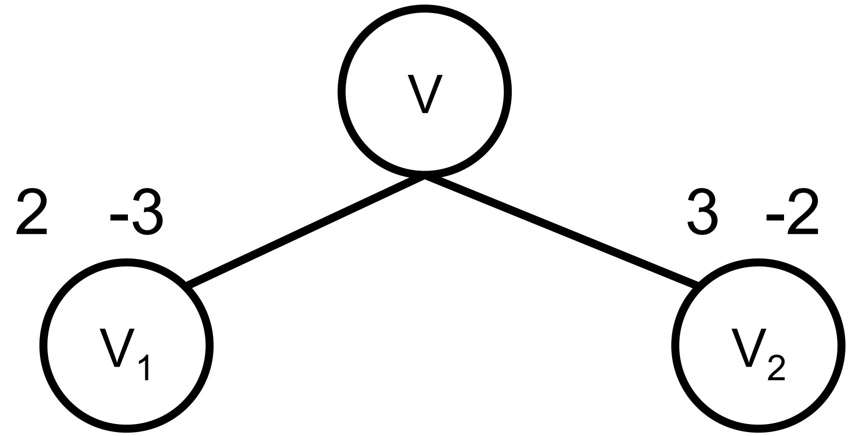


BOUND: Compute lower bounds for each branch

Convex relaxations (incomplete methods)

Branch-and-Bound

Find **v** ∈ V such that h(**v**) ≤ 0

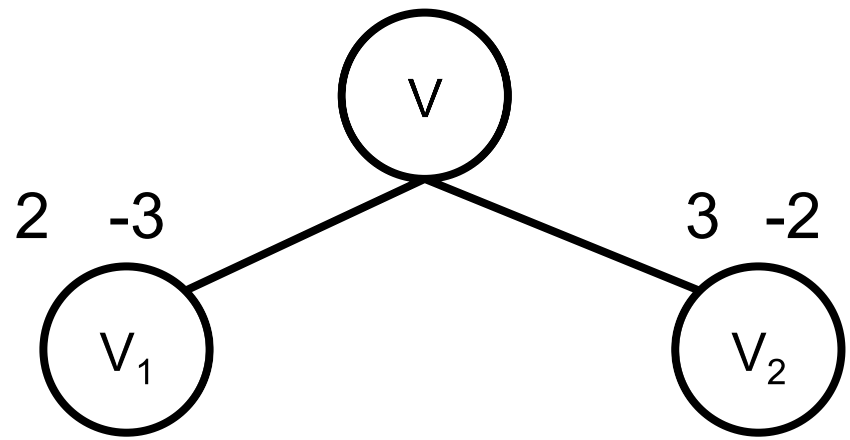


PRUNE: Any lower bounds greater than 0?

NO

Branch-and-Bound

Find **v** ∈ V such that $h(\mathbf{v}) \leq 0$

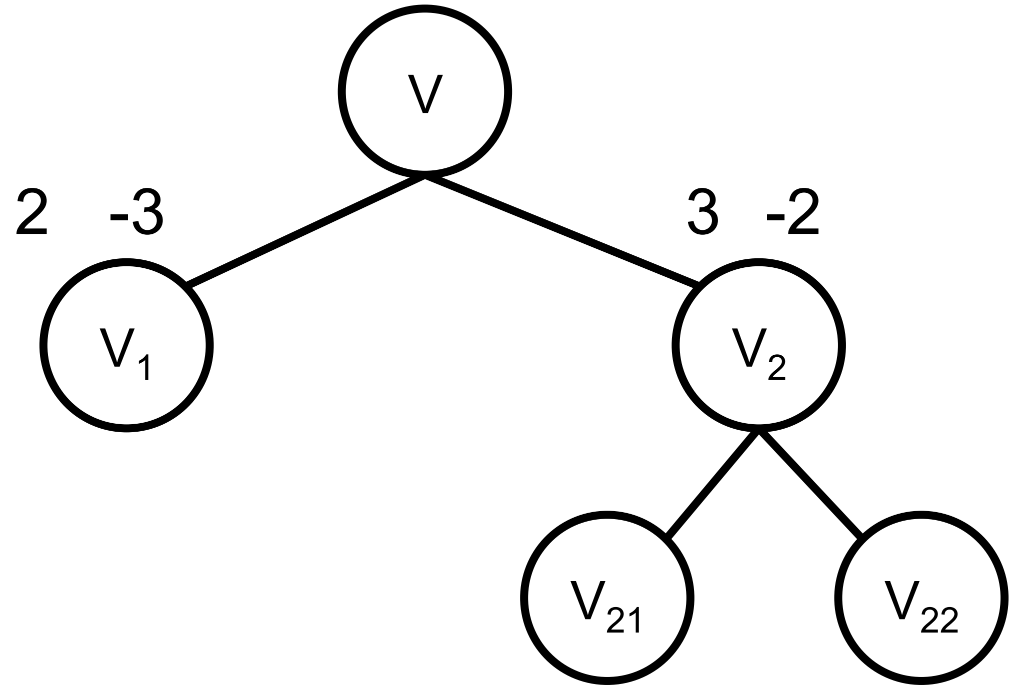


SELECT: Choose a subproblem

Say, we choose V_2

Branch-and-Bound

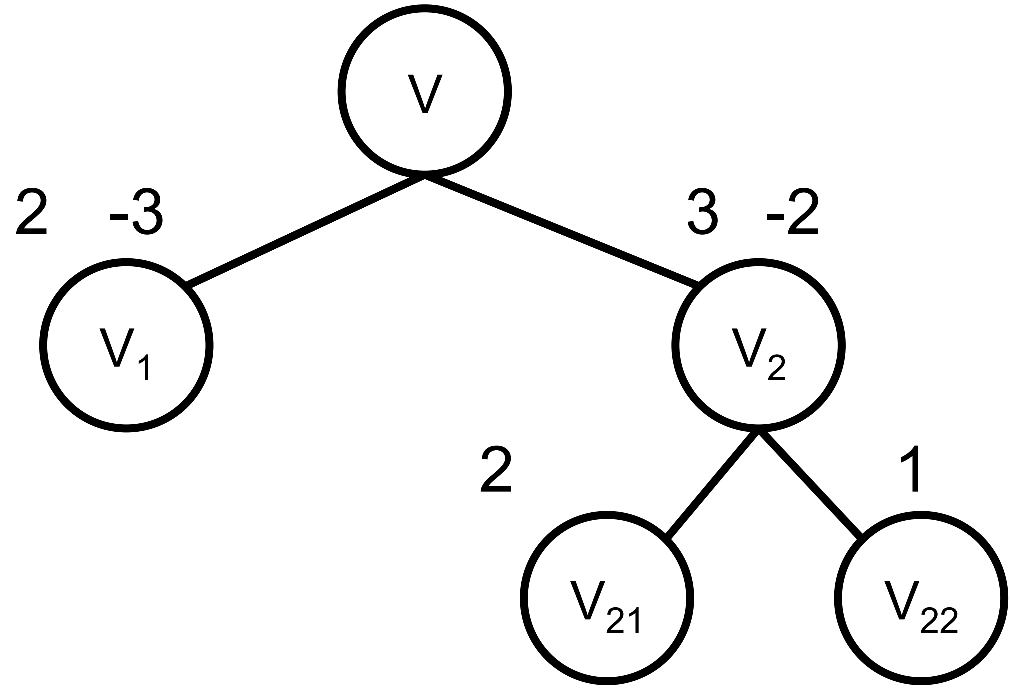
Find **v** ∈ V such that h(**v**) ≤ 0



BRANCH: Split the feasible set

Branch-and-Bound

Find **v** ∈ V such that $h(\mathbf{v}) \leq 0$

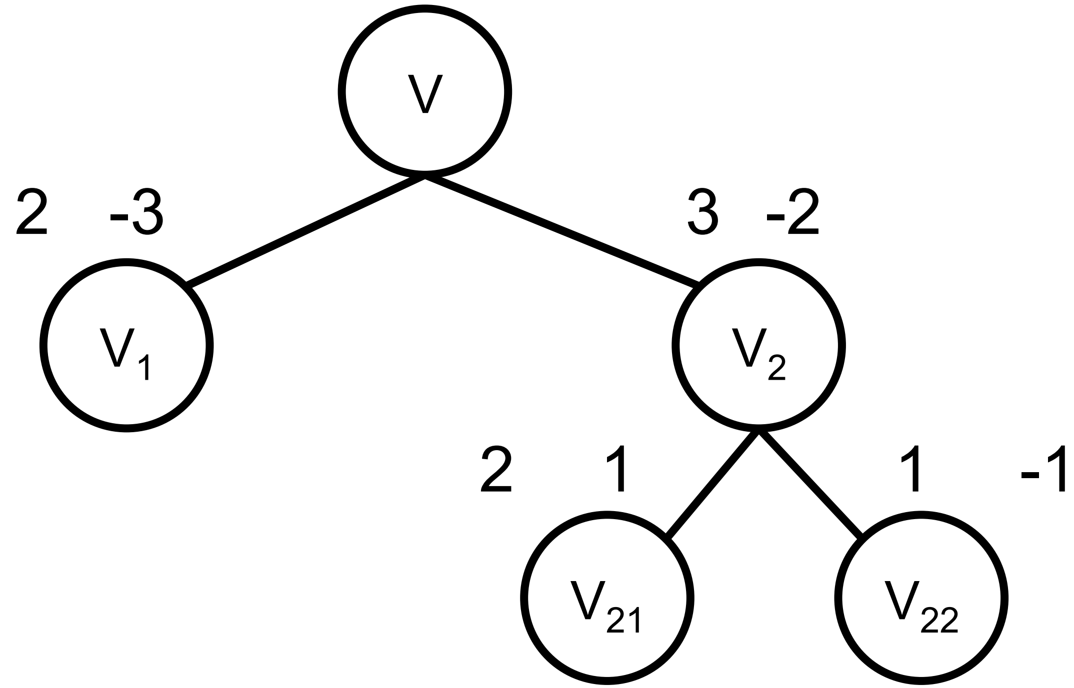


BOUND: Compute upper bounds

Upper bounds of children are smaller than the parent

Branch-and-Bound

Find **v** ∈ V such that h(**v**) ≤ 0

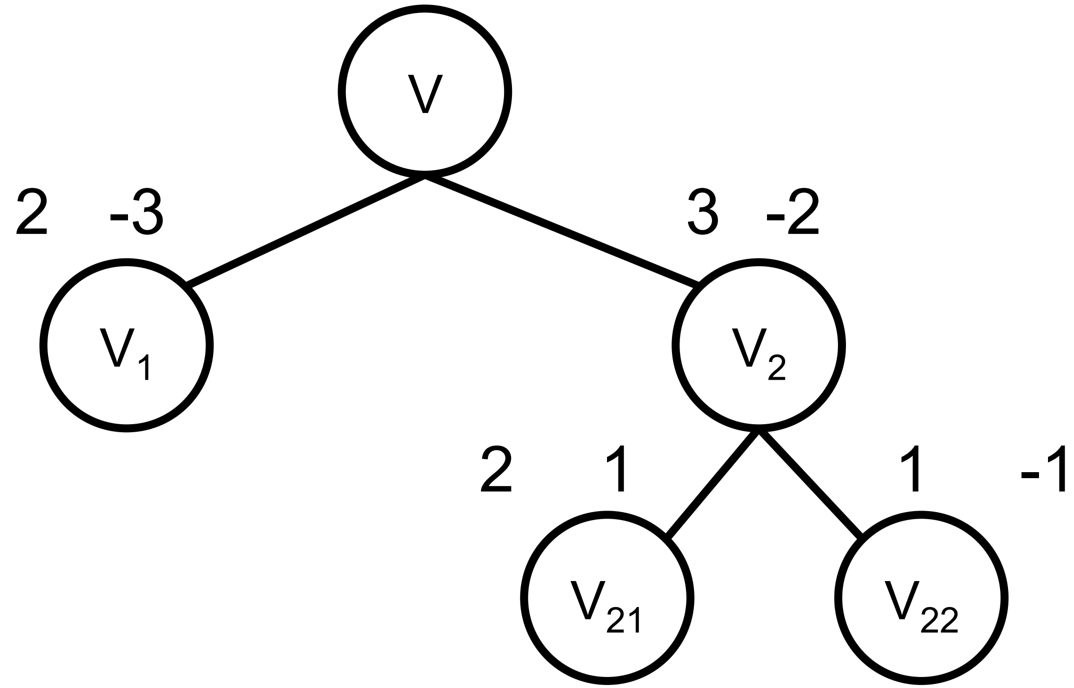


BOUND: Compute lower bounds

Lower bounds of children are greater than the parent

Branch-and-Bound

Find **v** ∈ V such that h(**v**) ≤ 0

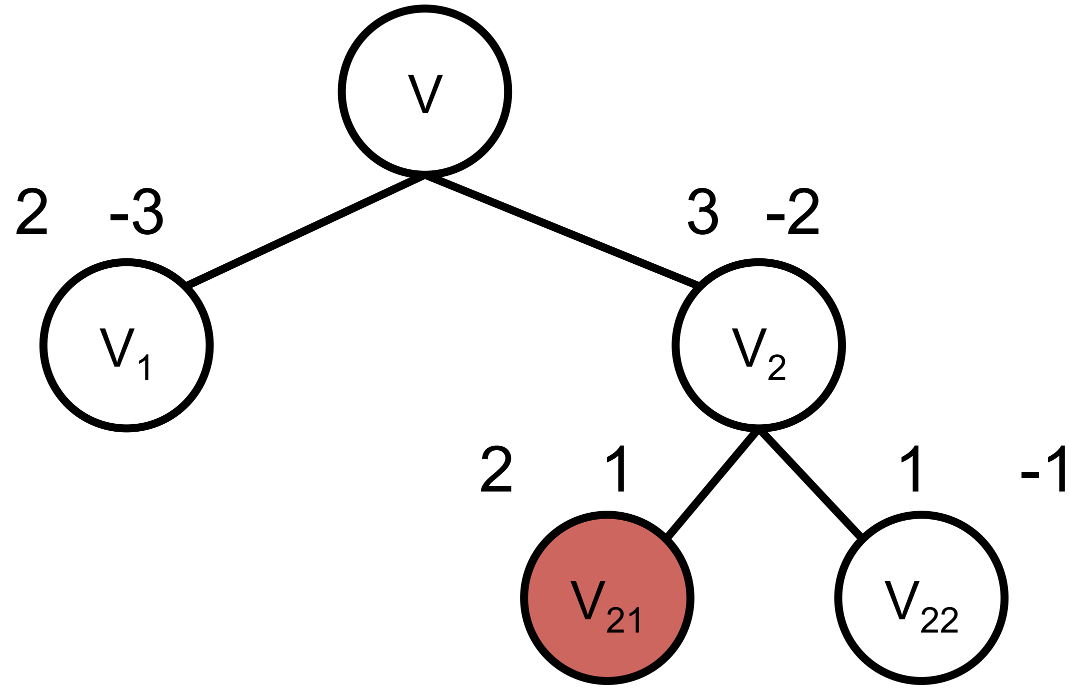


PRUNE: Any lower bounds greater than 0?

YES

Branch-and-Bound

Find **v** ∈ V such that h(**v**) ≤ 0

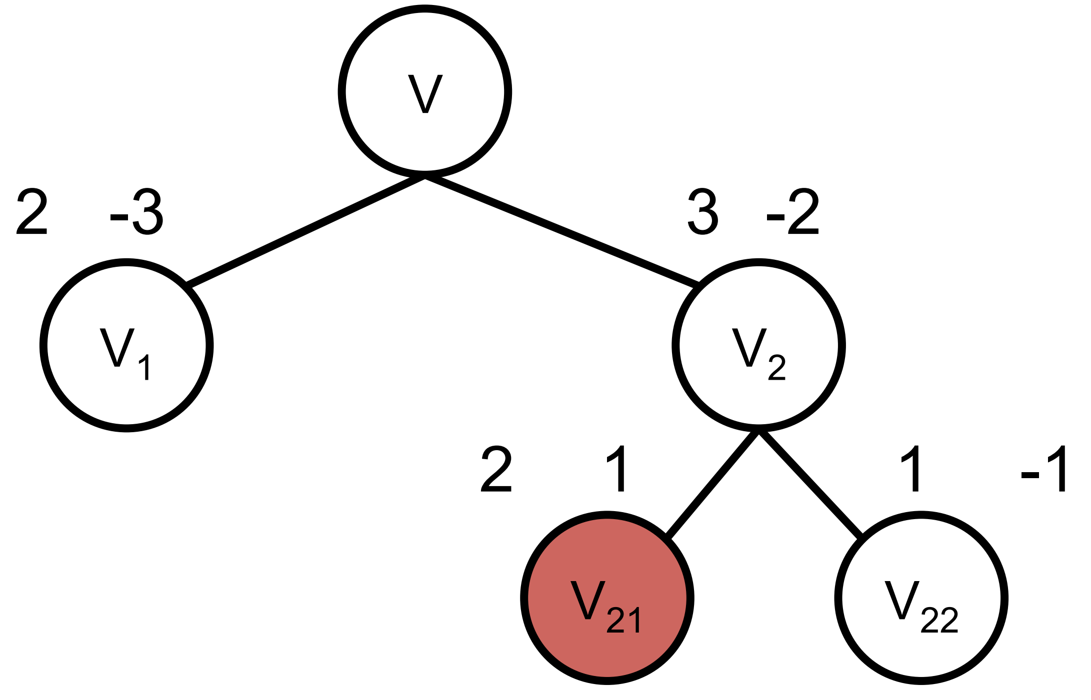


PRUNE: Any lower bounds greater than 0?

YES

Branch-and-Bound

Find **v** ∈ V such that h(**v**) ≤ 0

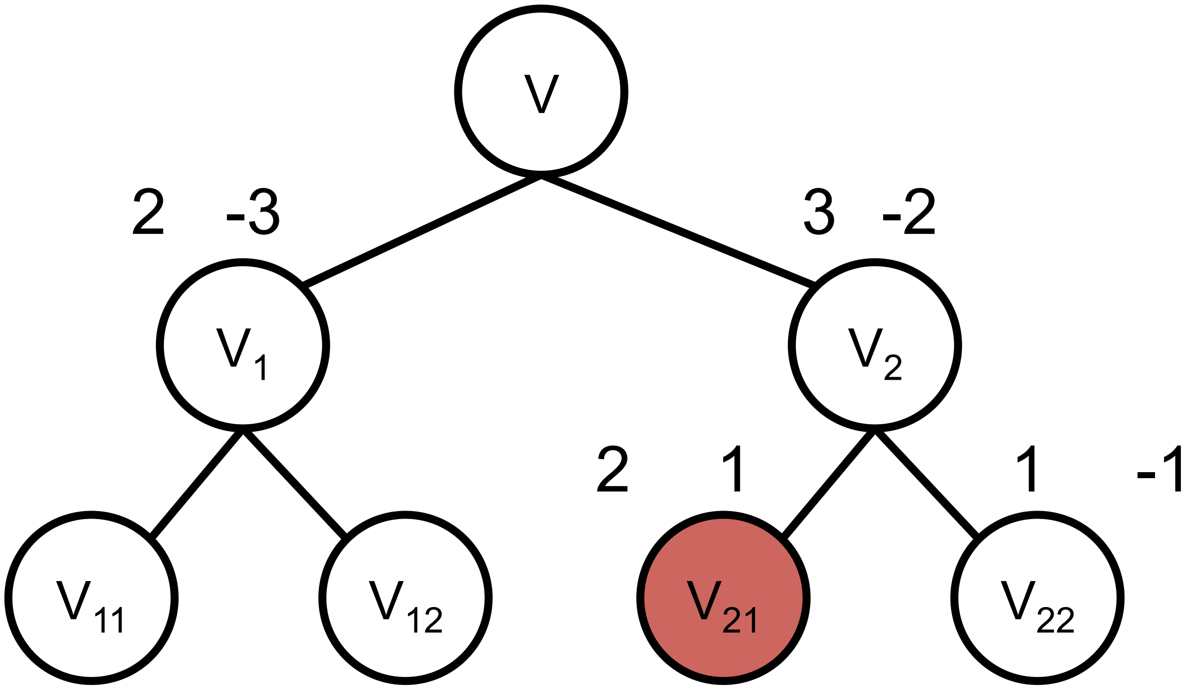


SELECT: Choose a subproblem

Say, we choose V_1

Branch-and-Bound

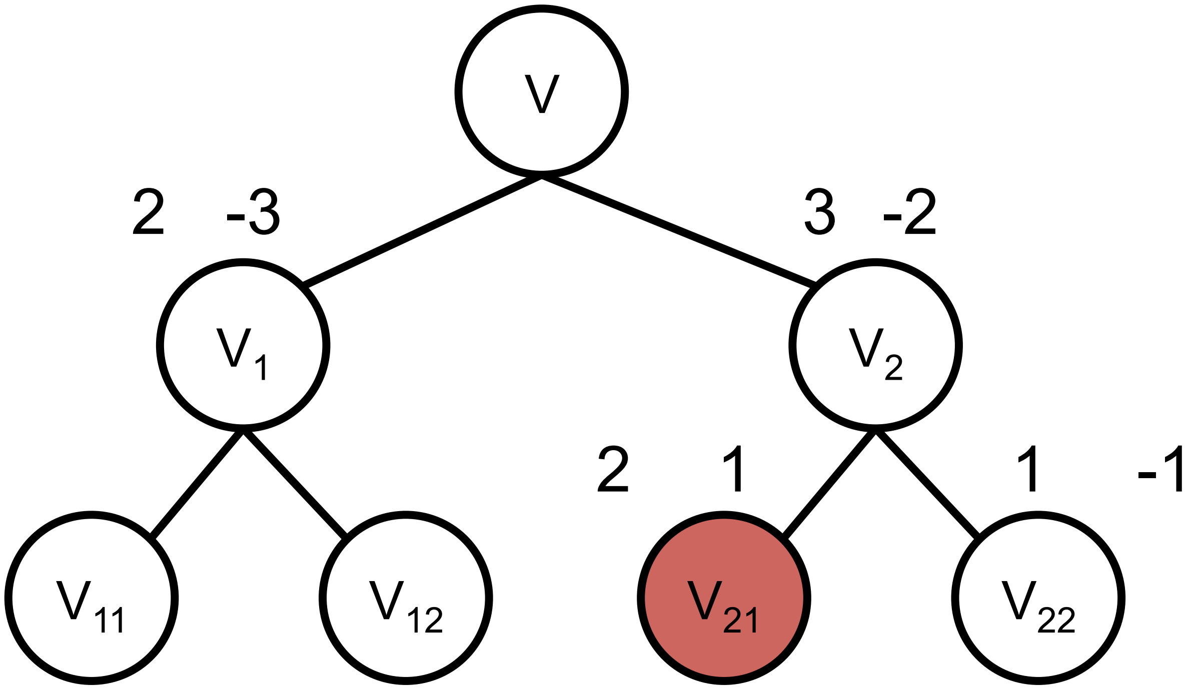
Find **v** ∈ V such that h(**v**) ≤ 0



BRANCH: Split the feasible set

Termination – Case I

Find $\mathbf{v} \in V$ such that $h(\mathbf{v}) \leq 0$

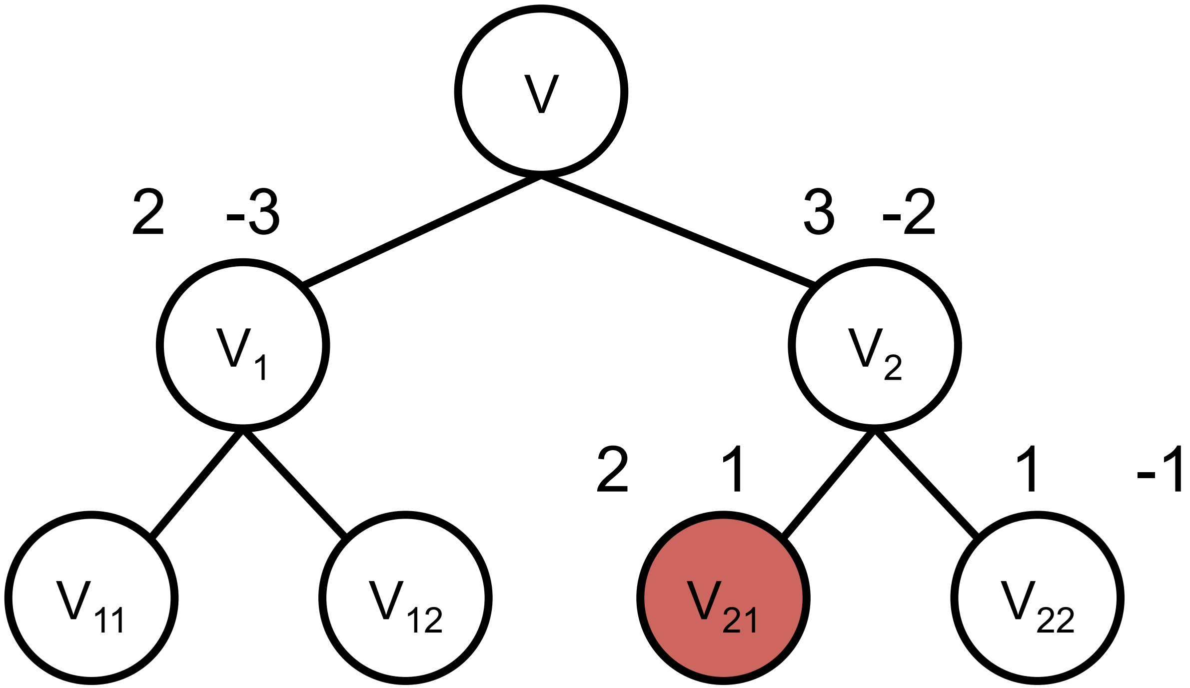


We find a counter-example

An upper bound that is less than 0

Termination – Case II

Find $\mathbf{v} \in V$ such that $h(\mathbf{v}) \leq 0$



We prove there does not exist $\mathbf{v} \in V$ s.t. $h(\mathbf{v}) \leq 0$

All leaf nodes have lower bound > 0

Unified Framework

All proposed complete methods are branch-and-bound

Choice of branching

Choice of bounding

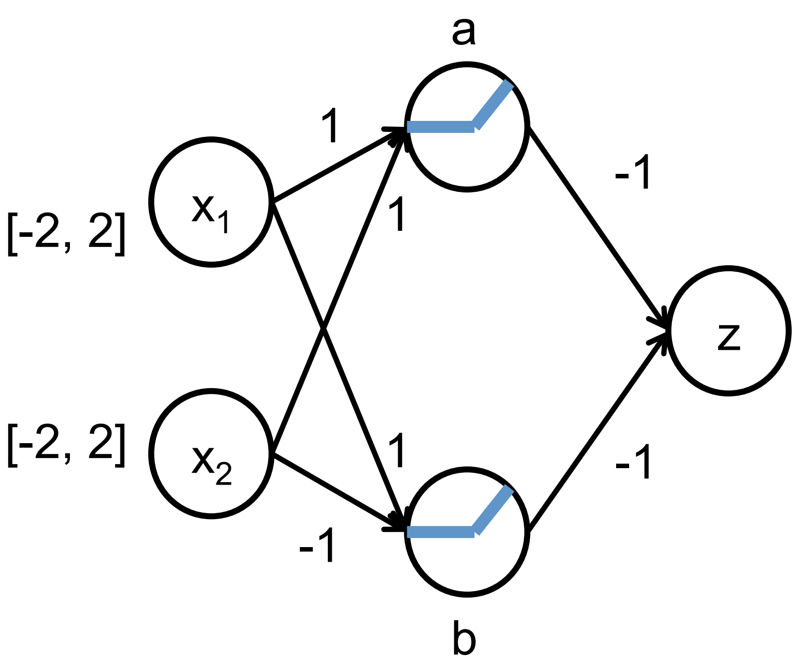
Choice of selecting (critical for false properties)

Outline

- **Reluplex**
- Planet
- Input Domain Branch-and-Bound
- Result

Katz, Barrett, Dill, Julian and Kochendefer, 2017

Example



Prove that z > -5

$-2 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} = \max\{a_{in}, 0\}$

$b_{out} = \max\{b_{in}, 0\}$

$z = - a_{out} - b_{out}$

$z \leq -5$

Bounding

x_1	x_2	a_{in}	a_{out}	b_{in}	b_{out}	z
0	0	0	0	0	0	0

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$b_{out} = \max\{b_{in}, 0\}$

$z = - a_{out} - b_{out}$

$z \leq -5$

Infeasible

Bounding

x_1	x_2	a_{in}	a_{out}	b_{in}	b_{out}	z
0	0	0	0	0	0	0

$$-2 \leq x_1 \leq 2$$

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$$a_{out} = \max\{a_{in}, 0\}$$

$$b_{out} = \max\{b_{in}, 0\}$$

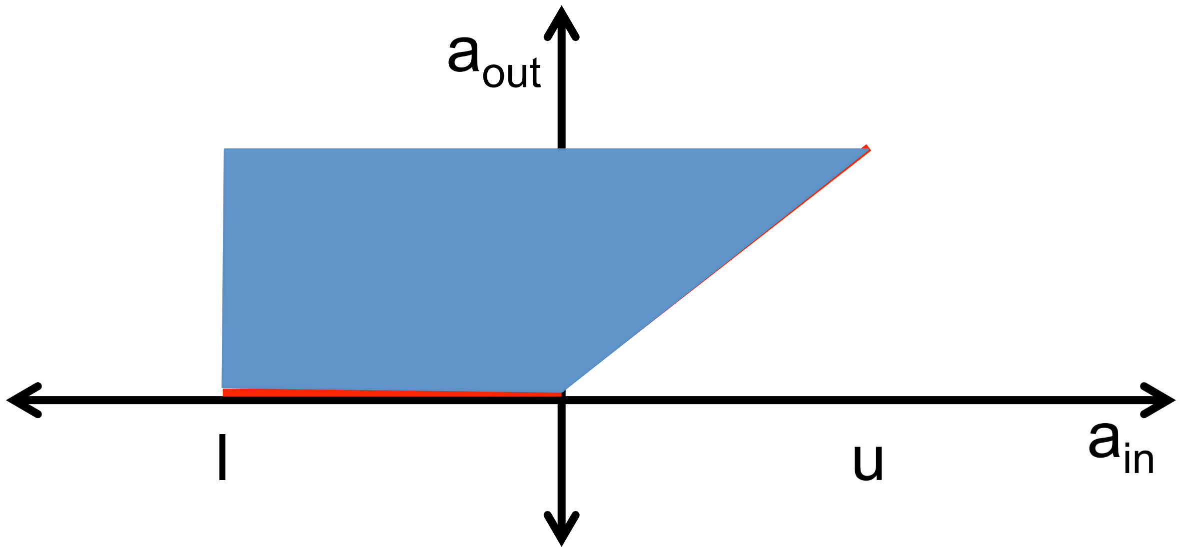
$$z = - a_{out} - b_{out}$$

$$z \leq -5$$

Relax all non-linearities

Relaxation

$a_{out} = \max\{a_{in}, 0\}$ $\qquad$ $a_{in} \in [l,u]$



Replace with convex superset

Bounding

x_1	x_2	a_{in}	a_{out}	b_{in}	b_{out}	z
0	0	0	0	0	0	0

$$-2 \leq x_1 \leq 2$$

$$-2 \leq x_2 \leq 2$$

$$a_{in} = x_1 + x_2$$

$$b_{in} = x_1 - x_2$$

$$a_{out} \geq a_{in},\ a_{out} \geq 0,\ a_{out} \leq 4$$

$$b_{out} \geq b_{in},\ b_{out} \geq 0,\ b_{out} \leq 4$$

$$z = -\ a_{out} - b_{out}$$

$$z \leq -5$$

Relax all non-linearities

Bounding

x_1	x_2	a_{in}	a_{out}	b_{in}	b_{out}	z
0	0	0	1	0	4	-5

$-2 \leq x_1 \leq 2$

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$a_{in} = x_1 + x_2$

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$a_{out} \geq a_{in}, \ a_{out} \geq 0, \ a_{out} \leq 4$

$b_{out} \geq b_{in}, \ b_{out} \geq 0, \ b_{out} \leq 4$

$z = - a_{out} - b_{out}$

$z \leq -5$

Bounding

x_1	x_2	a_{in}	a_{out}	b_{in}	b_{out}	z
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Infeasible

Bounding

x_1	x_2	a_{in}	a_{out}	b_{in}	b_{out}	z
0	0	0	1	0	4	-5

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$$z = - a_{out} - b_{out}$$

$$z \leq -5$$

Fix a non-linearity (say b)

Relax other non-linearities

Bounding

x_1	x_2	a_{in}	a_{out}	b_{in}	b_{out}	z
0	0	0	1	0	4	-5

$-2 \le x_1 \le 2$

$-2 \le x_2 \le 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} \ge a_{in},\ a_{out} \ge 0,\ a_{out} \le 4$

$b_{out} = b_{in}$

$z = - a_{out} - b_{out}$

$z \le -5$

Fix a non-linearity (say b)

Relax other non-linearities

Bounding

x_1	x_2	a_{in}	a_{out}	b_{in}	b_{out}	z
2	-2	0	1	4	4	-5

$-2 \leq x_1 \leq 2$

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$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

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Bounding

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$z = - a_{out} - b_{out}$

$z \leq -5$

Repeat

Try to fix some non-linearities

Relax other non-linearities

After some attempts, branch

Branching

Choose a non-linearity that is fixed many times

$-2 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

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$b_{out} = \max\{b_{in}, 0\}$

$z = - a_{out} - b_{out}$

$z \leq -5$

Branching

Choose a non-linearity that is fixed many times

Split into two

$-2 \le x_1 \le 2$

$-2 \le x_2 \le 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$\mathbf{a_{out} = max\{a_{in},0\}}$

$b_{out} = max\{b_{in},0\}$

$z = - a_{out} - b_{out}$

$z \le -5$

Branching

$-2 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} = 0$

$b_{out} = \max\{b_{in}, 0\}$

$z = - a_{out} - b_{out}$

$z \leq -5$

$-2 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} = a_{in}$

$b_{out} = \max\{b_{in}, 0\}$

$z = - a_{out} - b_{out}$

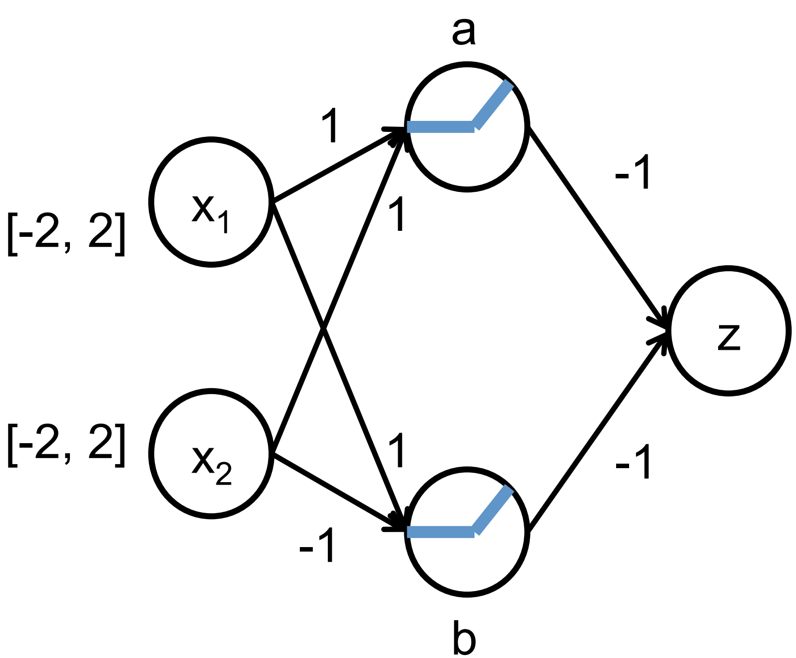
$z \leq -5$

Outline

- Reluplex
- **Planet**
- Input Domain Branch-and-Bound
- Result

Ehlers, 2017

Example



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Relax all non-linearities

$z \le -5$

Relaxation

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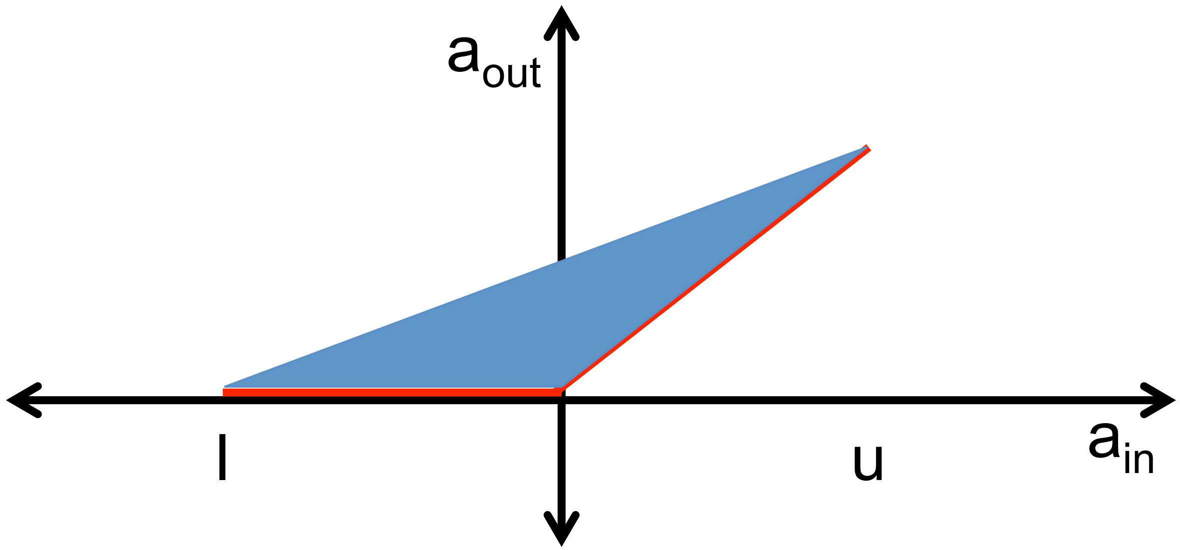

Replace with convex superset

Bounding

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$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

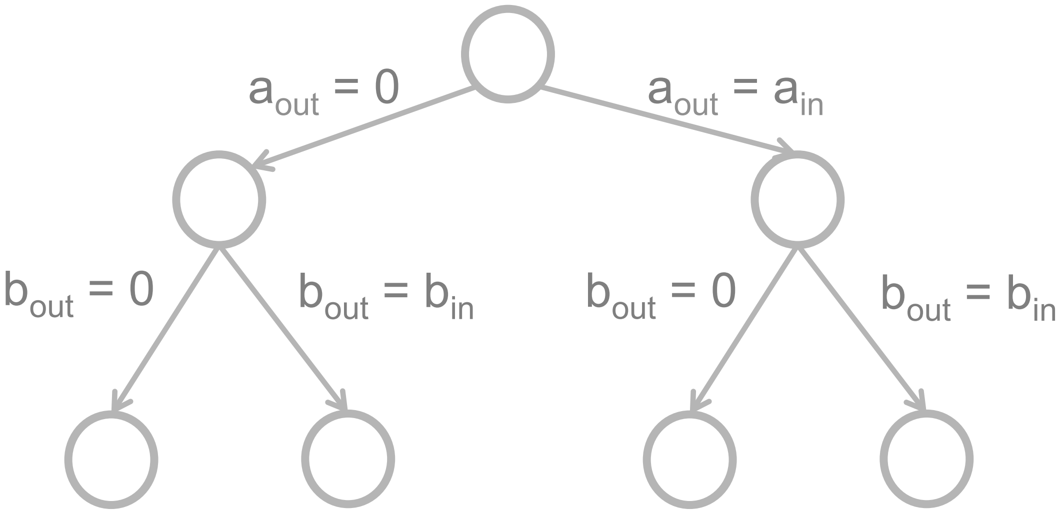
$a_{out} \geq a_{in}$, $a_{out} \geq 0$, $a_{out} \leq a_{in}/2+2$

$b_{out} \geq b_{in}$, $b_{out} \geq 0$, $b_{out} \leq b_{in}/2+2$

$z = - a_{out} - b_{out}$

$z \leq -5$

Branching



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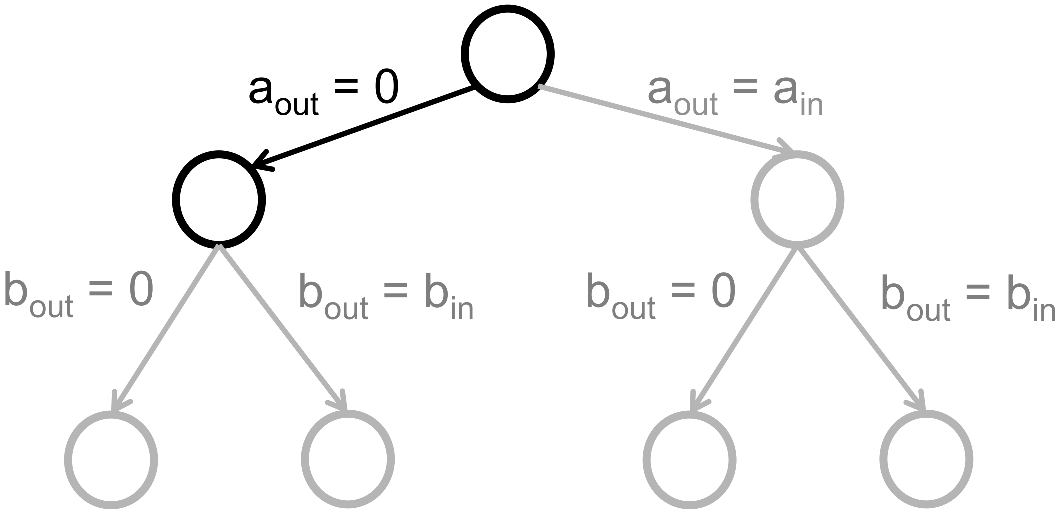
Branching



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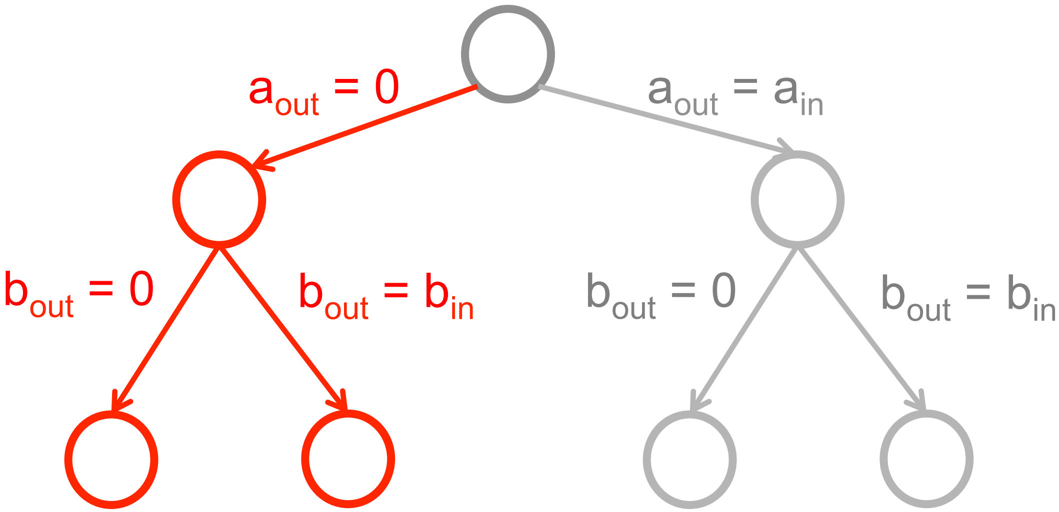
$z = -\ a_{out} - b_{out}$

$z \leq -5$

Feasible? NO

Branching



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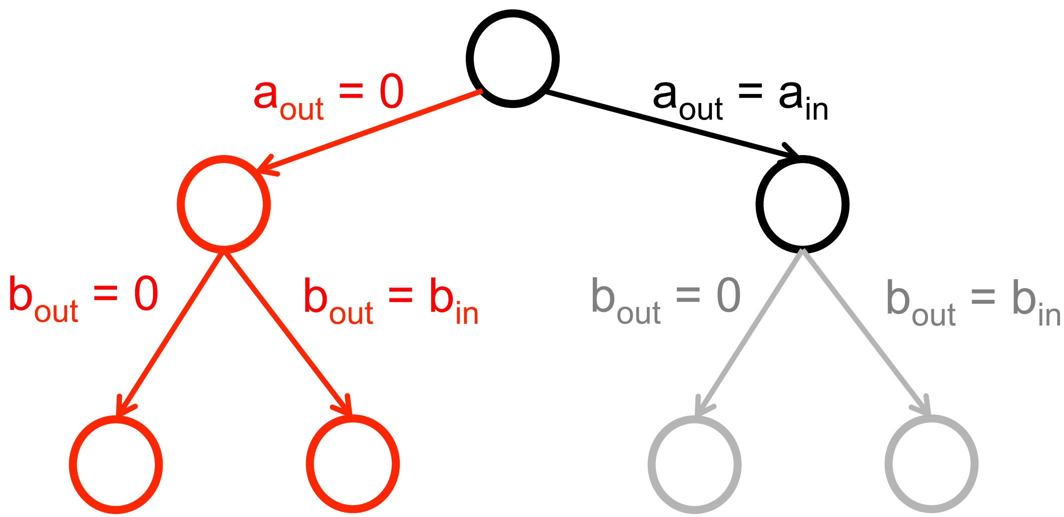
$b_{out} \geq b_{in},\ b_{out} \geq 0,\ b_{out} \leq b_{in}/2+2$

$z = -a_{out} - b_{out}$

$z \leq -5$

Prune away

Branching



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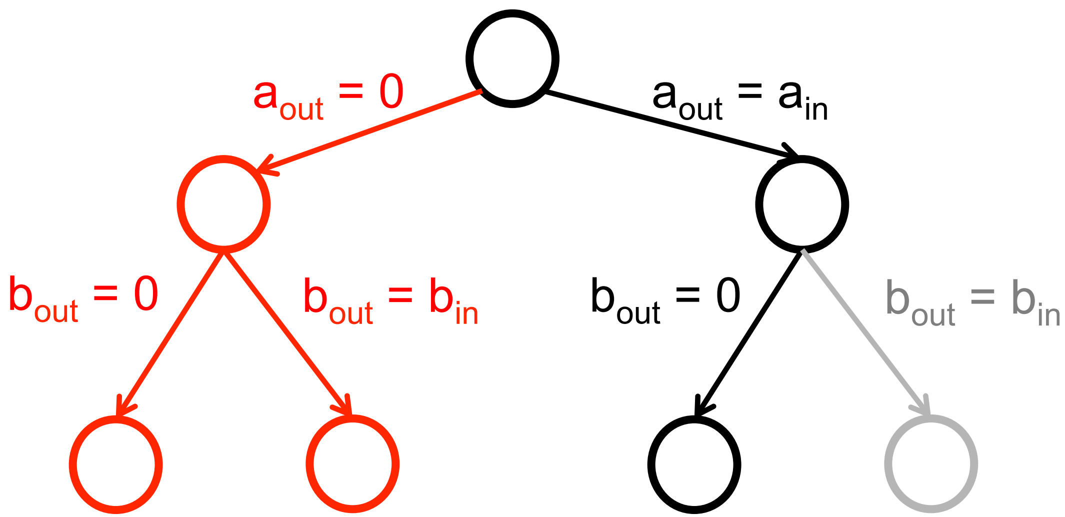
$z = -\ a_{out} - b_{out}$

$z \leq -5$

Feasible? YES

$x_1 = 2,\ x_2 = 2,\ a_{out} = 4,\ b_{out} = 2$

Branching



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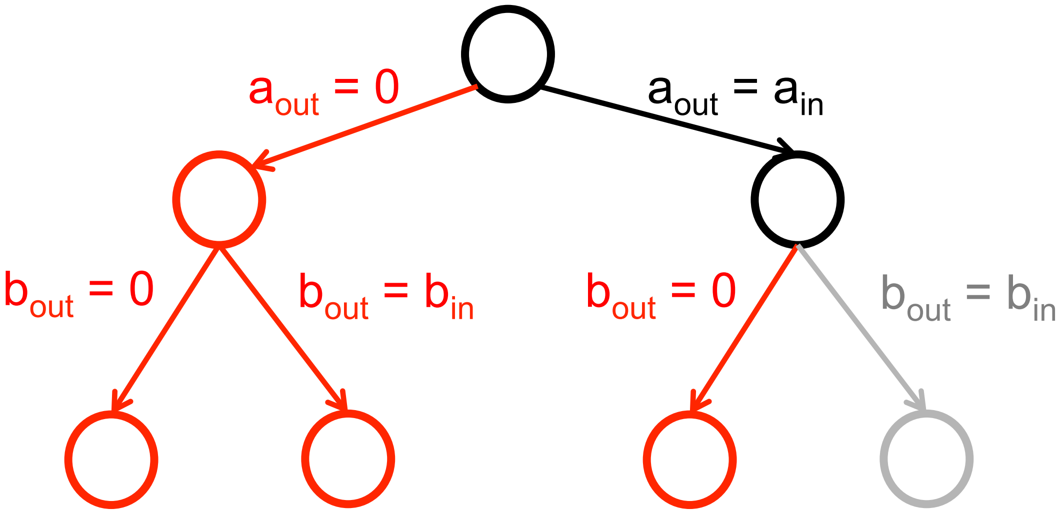
Feasible? NO

Branching



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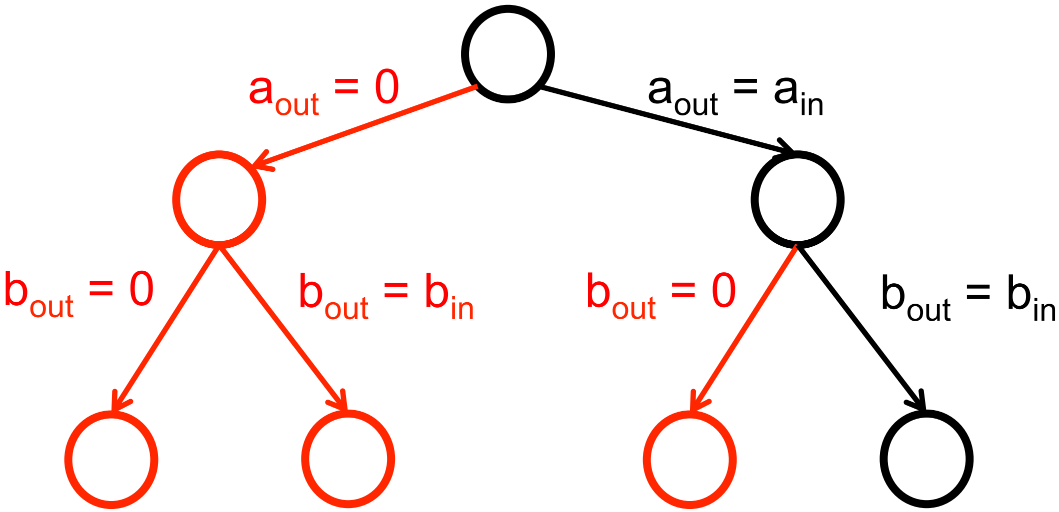
$$z = - a_{out} - b_{out}$$

$$z \leq -5$$

Prune away

Branching



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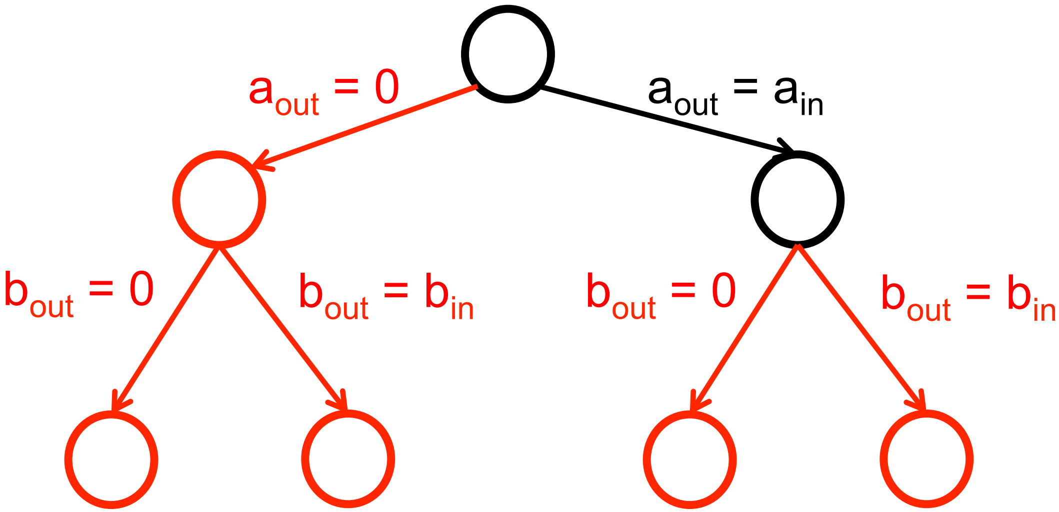
$$z \leq -5$$

Feasible? NO

Branching



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$z = - a_{out} - b_{out}$

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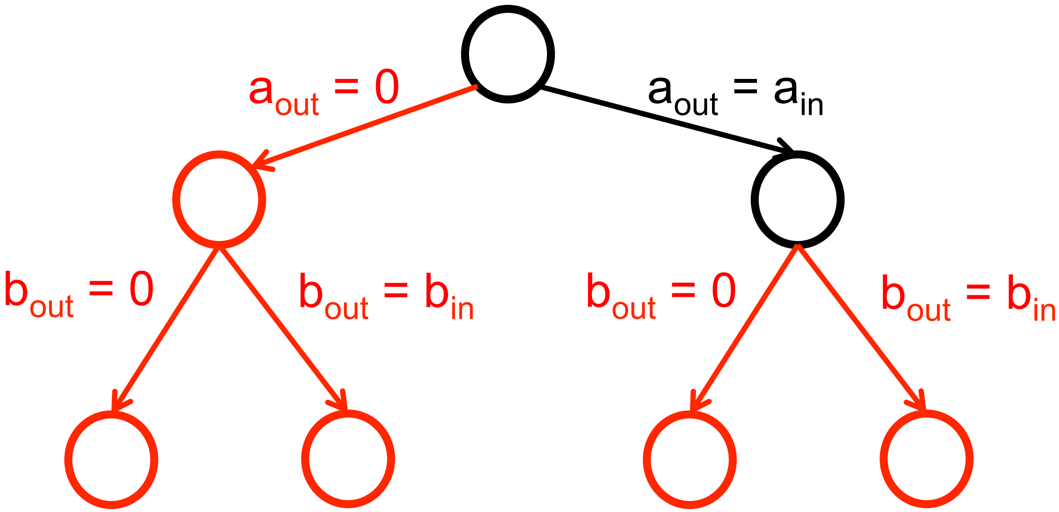
Prune away

Branching



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$$z = -\ a_{out} - b_{out}$$

Property must be true

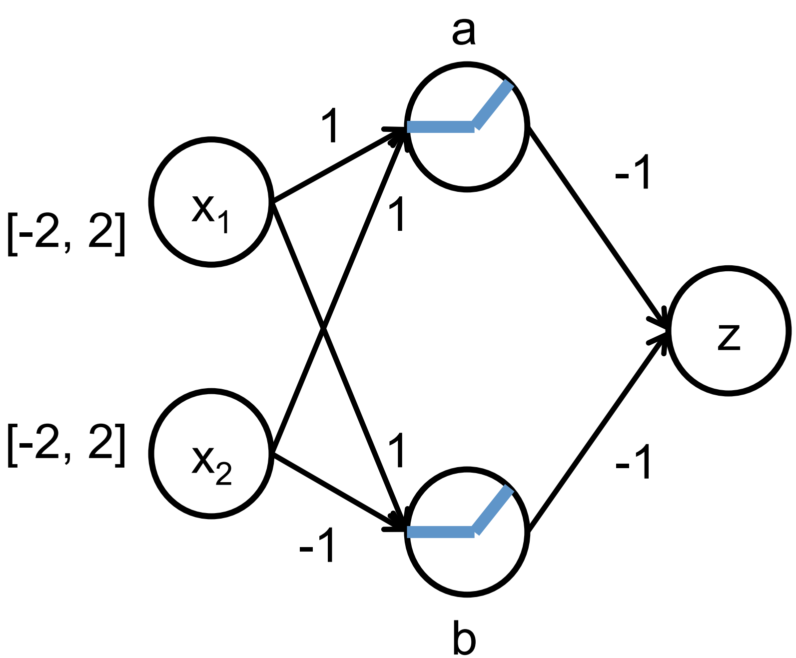
$$z \leq -5$$

Outline

- Reluplex
- Planet
- **Input Domain Branch-and-Bound**
- Result

Bunel, Turkaslan, Torr, Kohli and Kumar, 2018

Example



Prove that $z > -5$

$-2 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} = \max\{a_{in}, 0\}$

$b_{out} = \max\{b_{in}, 0\}$

$z = - a_{out} - b_{out}$

Bounding

$-2 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} = \max\{a_{in},0\}$

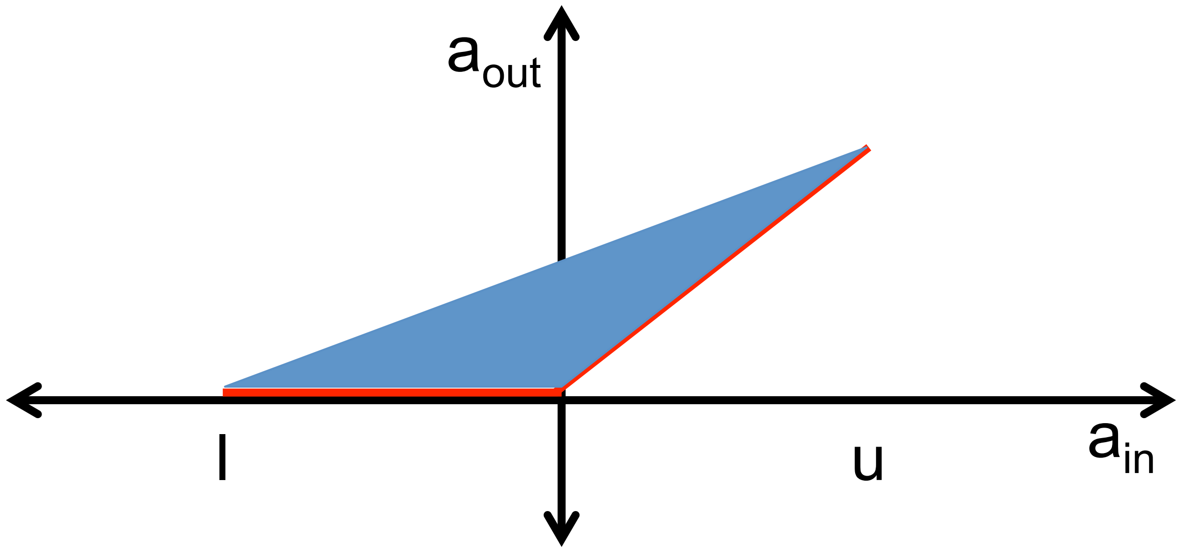
$b_{out} = \max\{b_{in},0\}$

$z = - a_{out} - b_{out}$

Relax all non-linearities

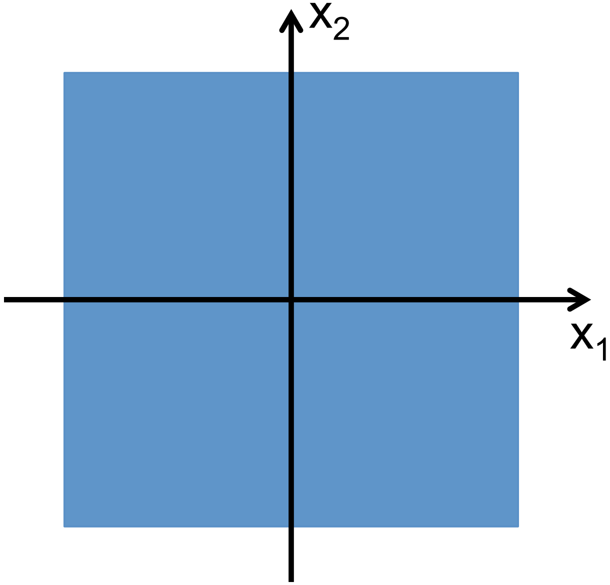
Relaxation

$a_{out} = \max\{a_{in}, 0\}$ $\qquad$ $a_{in} \in [l,u]$



Replace with convex superset

Bounding



$-2 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} \geq a_{in},\ a_{out} \geq 0,\ a_{out} \leq a_{in}/2+2$

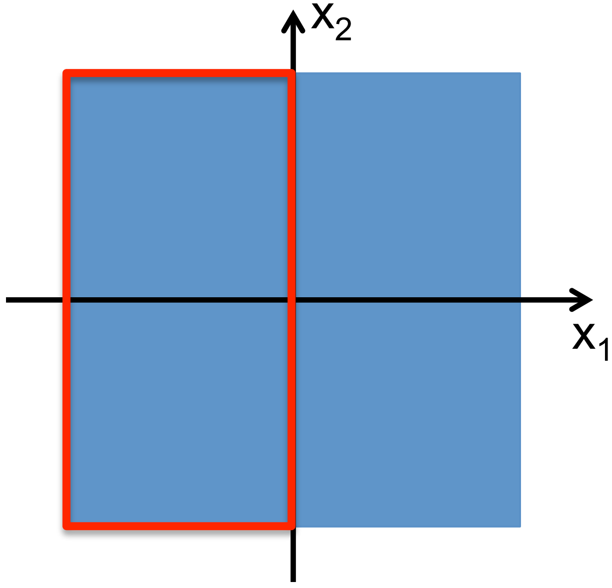
$b_{out} \geq b_{in},\ b_{out} \geq 0,\ b_{out} \leq b_{in}/2+2$

$z = - a_{out} - b_{out}$

$z_{min} = -6$

min z

Bounding



$-2 \le x_1 \le 2$

$-2 \le x_2 \le 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

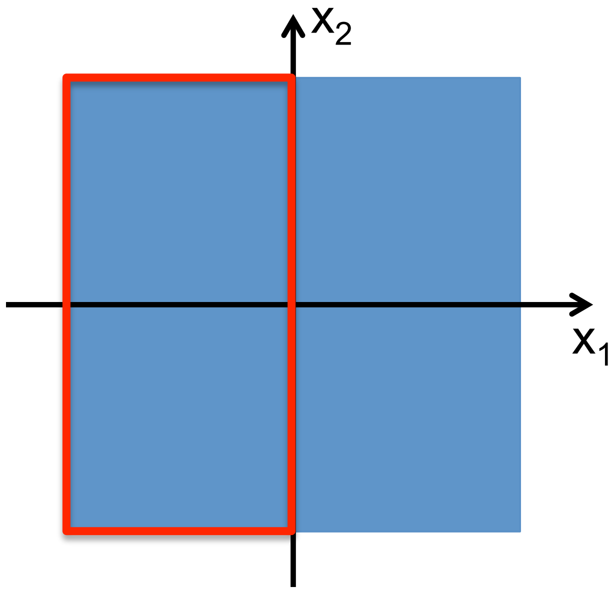
$a_{out} \ge a_{in}$, $a_{out} \ge 0$, $a_{out} \le a_{in}/2+2$

$b_{out} \ge b_{in}$, $b_{out} \ge 0$, $b_{out} \le b_{in}/2+2$

$z = - a_{out} - b_{out}$

$\min z$

Bounding



$-2 \le x_1 \le 0$

$-2 \le x_2 \le 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} \ge a_{in}$, $a_{out} \ge 0$, $a_{out} \le a_{in}/3+4/3$

$b_{out} \ge b_{in}$, $b_{out} \ge 0$, $b_{out} \le b_{in}/3+4/3$

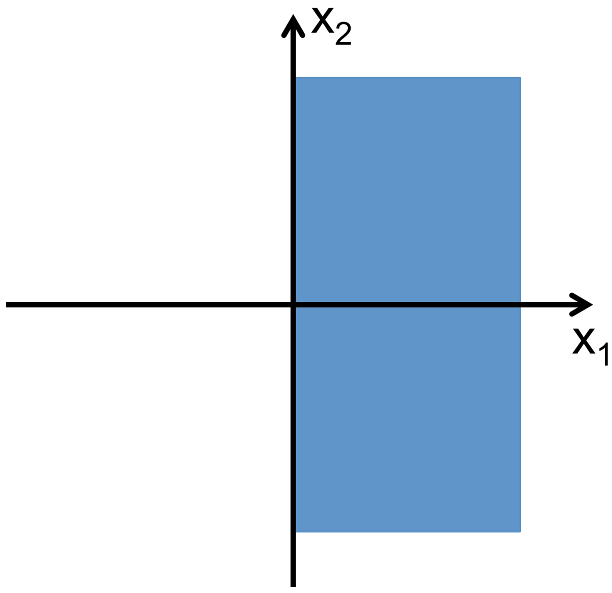
Prune away

$z = - a_{out} - b_{out}$

$z_{min} = -2.66$

min z

Bounding



$0 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} \geq a_{in}$, $a_{out} \geq 0$, $a_{out} \leq 2a_{in}/3+4/3$

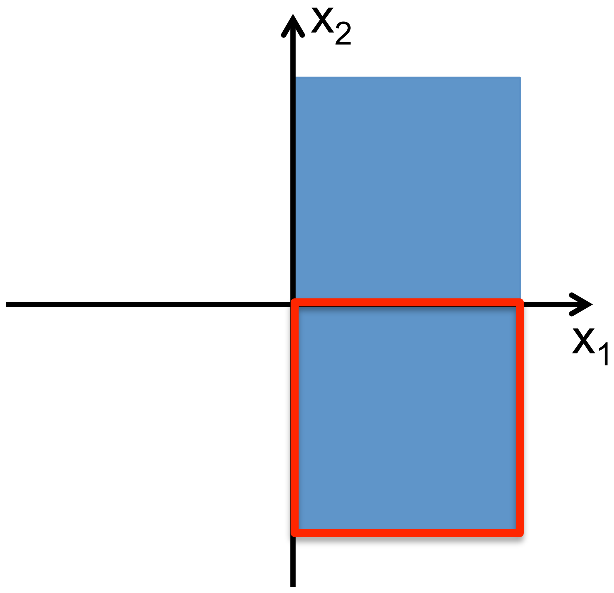
$b_{out} \geq b_{in}$, $b_{out} \geq 0$, $b_{out} \leq 2b_{in}/3+4/3$

$z = - a_{out} - b_{out}$

min z

$z_{min} = -5.33$

Bounding



$0 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 2$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

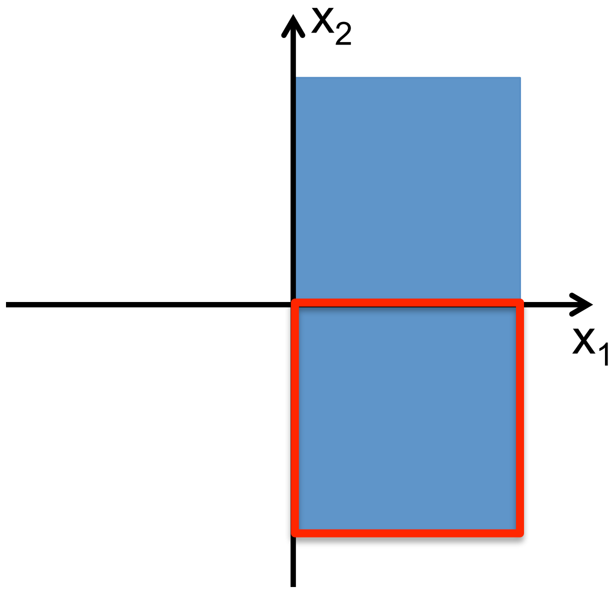
$a_{out} \geq a_{in}$, $a_{out} \geq 0$, $a_{out} \leq 2a_{in}/3+4/3$

$b_{out} \geq b_{in}$, $b_{out} \geq 0$, $b_{out} \leq 2b_{in}/3+4/3$

$z = - a_{out} - b_{out}$

min z

Bounding



$0 \leq x_1 \leq 2$

$-2 \leq x_2 \leq 0$

$a_{in} = x_1 + x_2$

$b_{in} = x_1 - x_2$

$a_{out} \geq a_{in},\ a_{out} \geq 0,\ a_{out} \leq a_{in}/2+1$

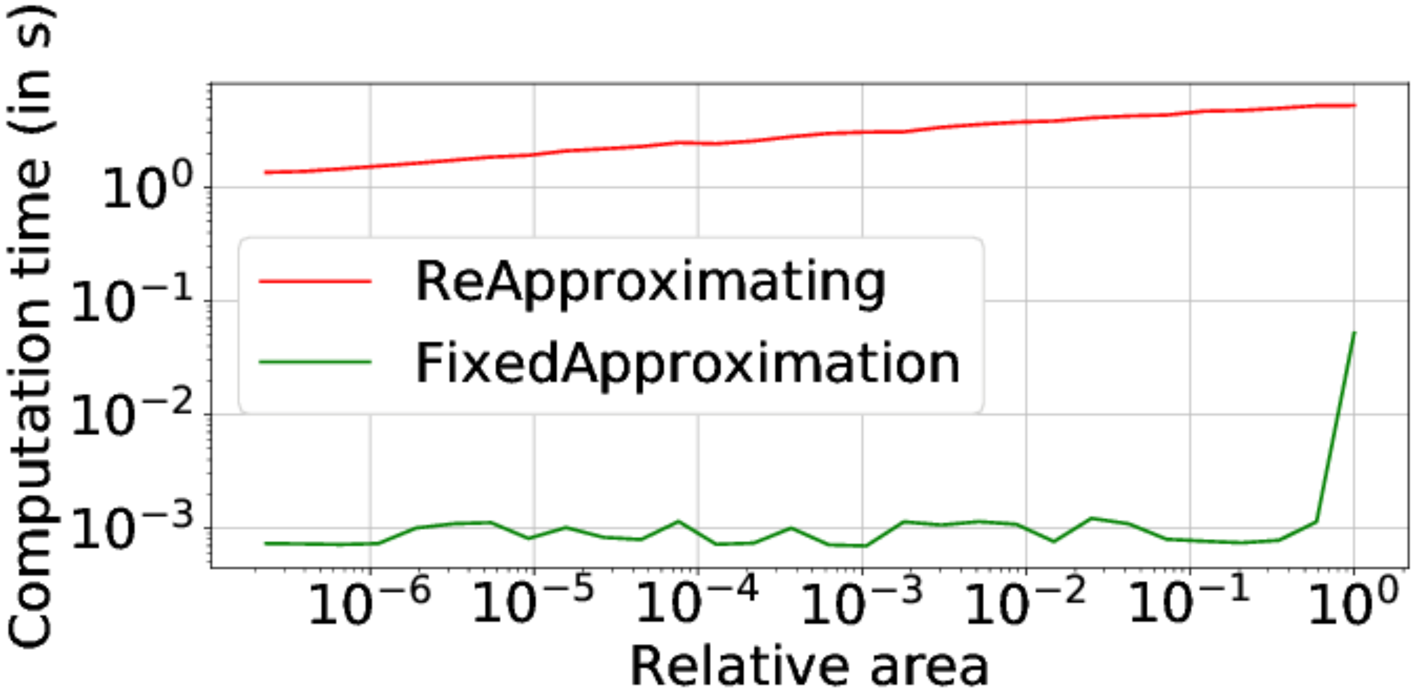
$b_{out} \geq b_{in},\ b_{out} \geq 0,\ b_{out} \leq b_{in}$

$z = -\ a_{out} - b_{out}$

Continue until termination

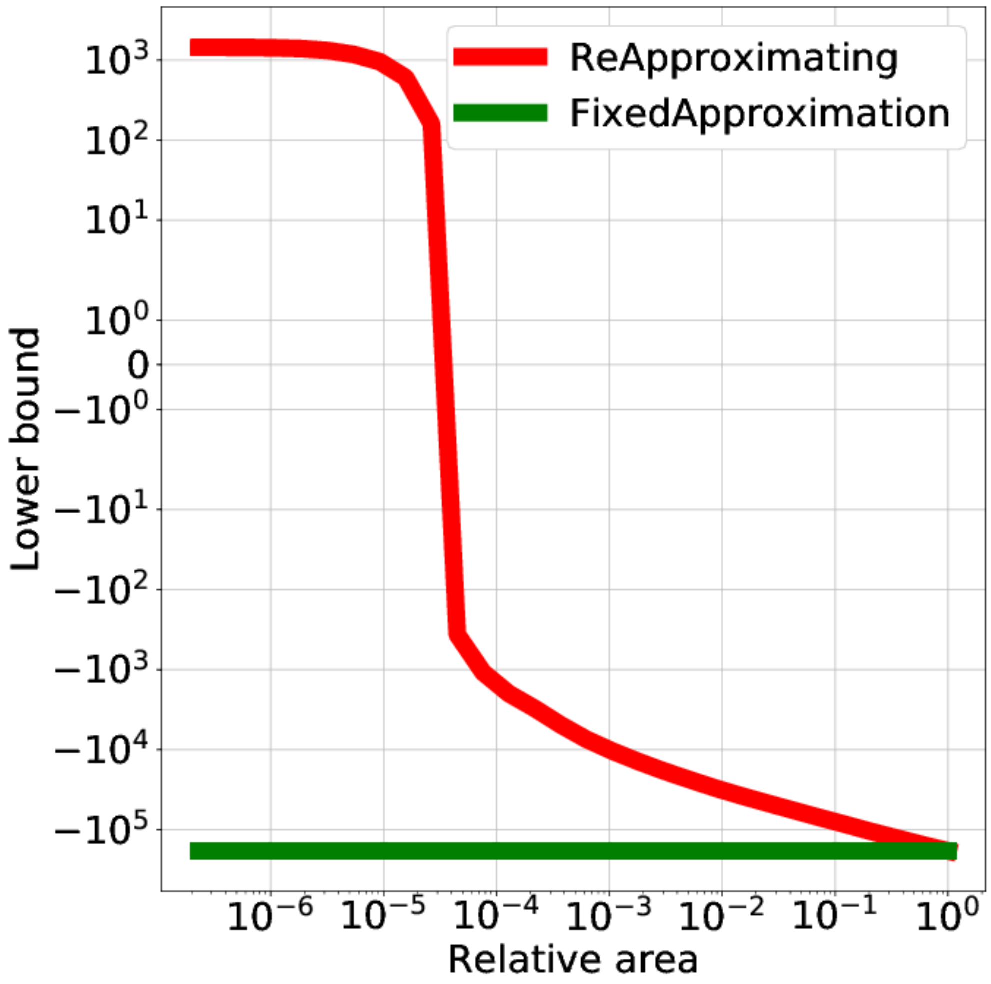
min z

Recomputing Bounds



Recomputing requires more computational time

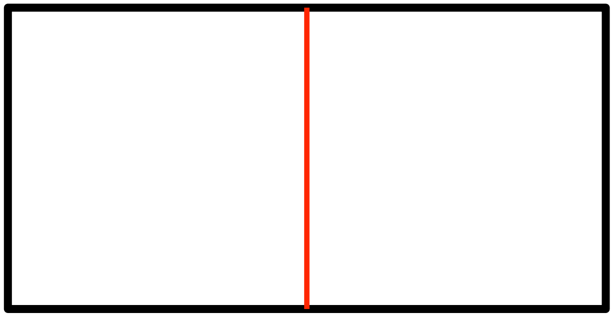
Recomputing Bounds



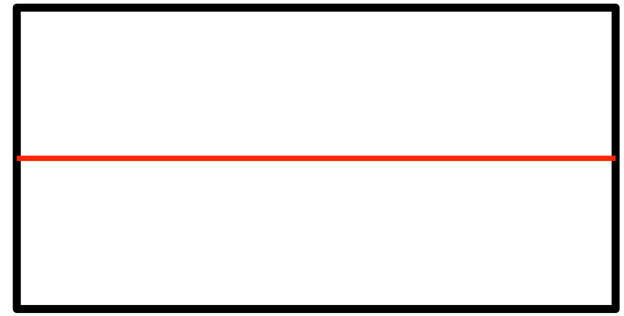
Recomputing provides significantly better bounds

Branching Choices

Option 1



Option 2



Compute efficient but approximate lower bounds

Wong and Kolter, 2018

Pick the option that maximizes lower bound

Larger lower bounds encourage pruning

Outline

- Reluplex
- Planet
- Input Domain Branch-and-Bound
- **Result**

ACAS Data Set

Publicly available data set

Katz et al., 2017

188 properties to verify

5 inputs, 6 hidden layers of 50 units

5 outputs corresponding to aircraft maneuvers

Results

Black box solvers verify ~20% properties before timeout

Cheng et al., 2017; Tjeng et al., 2017

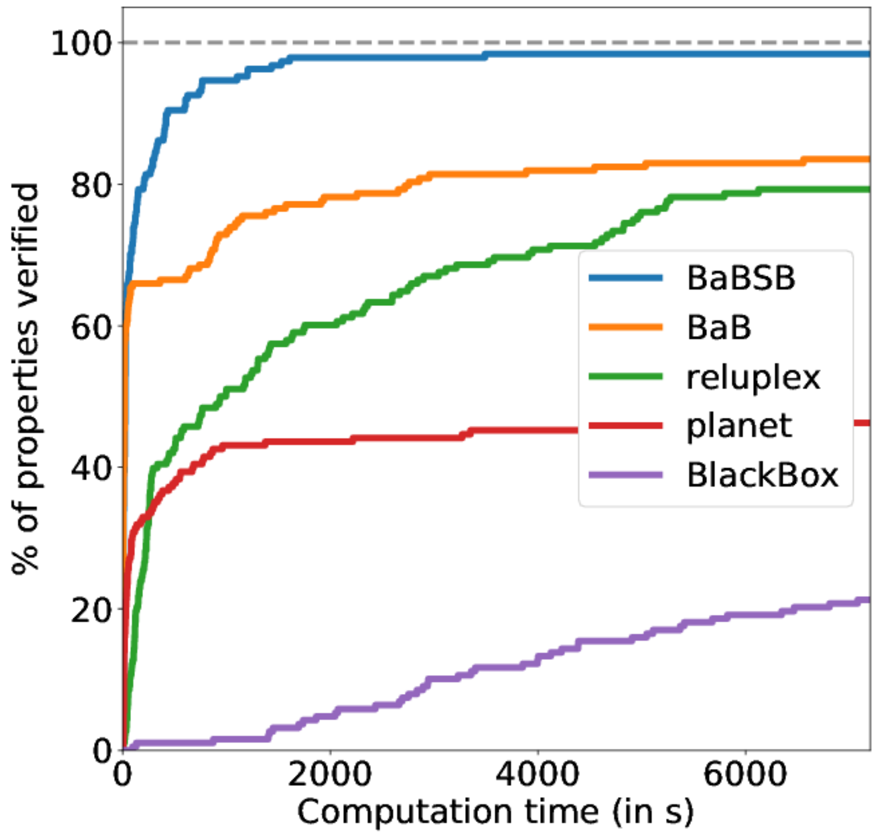
Previous branch-and-bound verifies ~40% properties

Ehlers 2017; Katz et al., 2017

Improved algorithm verifies almost all properties

Two orders of magnitude speed-up

Results



Ours

Katz et al., 2017

Ehlers et al., 2017

Black box

Questions?